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ON THE SCHNIRELMANN DENSITY OF THE k-FREE AND (k,r)-FREE INTEGERS

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled ON THE SCHNIRELMANN DENSITY OF THE k-FREE AND (k,r)-FREE INTEGERS submitted by GEORGE EUGENE HARDY in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

Date ....April ...

DEDICATION

This thesis is dedicated to my parents,

Mitchell and Marjorie Hardy

ABSTRACT

In this thesis, we investigate the Schnirelmann density of the k-free and of the (k,r)-free integers (which include, as special cases both k-free and unitarily k-free integers). We find the asymptotic density of the (k,r)-free integers and prove that the Schnirelmann and asymptotic densities of these classes of integers are different. We estimate this difference for the special case of the k-free integers. We determine (theoretically and via computer investigation) as precisely as practicable where these Schnirelmann densities are achieved. Related density results for the (k,r) integers are obtained.

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LIST OF NOTATIONS

Except where explicitly stated otherwise, the following notations are used as stated below:

$c_{(k;r)} = \prod_{p} (1 - p^{-r} + p^{-k})$

d_k = the Schnirelmann density of the r-free integers

$d_{k,r}$ = the Schnirelmann density of the (k,r) integers

$d_{(k;r)}$ = the Schnirelmann density of the (k,r)-free integers

D_r = the asymptotic density of the r-free integers

$D_{k,r}$ = the asymptotic density of the (k,r) integers

$D_{(k;r)}$ = the asymptotic density of the (k,r)-free integers

$E_k = D_k - (1 - 2^{-r} - 3^{-r} - 5^{-r})$

$E_k(n) = q_k(n)\cdot n^{-1} - (1 - 2^{-r} - 3^{-r} - 5^{-r})$

$E_{k,r}(x) = q_{k,r}(x) - x\cdot D_{k,r}$

$E_{(k;r)}(x) = q_{(k;r)}(x) - x\cdot D_{(k;r)}$

$F_k(n) = E_k(n) \cdot n \cdot 30^k$

$H = 120^a+60^a+40^a+30^a+24^a + (\frac{a+5}{a-6})\ 20^a$

$J = 420^a+210^a+140^a+105^a+84^a+70^a + (\frac{a+6}{a-1})\ 60^a$

$M(x) = \sum_{n \leq x} \mu(n)$

$M(x,m) = \sum_{\substack{n \leq m \\ (n,m)=1}} \mu(n)$

$M_m(n) = n - R_m(n)$

n_k = the smallest n on $[\frac{1}{2} \cdot 6^k, 6^k)$ at which $d_k = q_k(n)n^{-1}$

$n_{k,r}$ = any n such that $d_{k,r} = q_{k,r}(n) \cdot n^{-1}$

$n_{(k;r)}$ = any n such that $d_{(k;r)} = q_{(k;r)}(n) \cdot n^{-1}$

p = any prime number

$q_r(x)$ = the number of r-free integers $\leq x$

$q_{k,r}(x)$ = the number of (k,r) integers $\leq x$

$q_{(k;r)}(x)$ = the number of (k,r)-free integers $\leq x$

Q_r = the set of r-free integers

$Q_{k,r}$ = the set of (k,r) integers

$Q_{(k;r)}$ = the set of (k,r)-free integers

$R_m(n) = [n/m] \cdot m$

γ = Euler's constant

$\gamma(m)$ = the core of m = the largest square-free divisor of m

ζ = the Riemann Zeta function

$\mu(n)$ = the Möbuis μ function

$\phi_s(n) = n^s \prod_{p|n} 1 - p^{-s}$ = the generalized Jordan totient function

$$\chi_r(n) = \begin{cases} 0, & n \notin Q_r \\ 1, & n \in Q_r \end{cases}$$

$$\chi_{k,r}(n) = \begin{cases} 0, & n \notin Q_{k,r} \\ 1, & n \in Q_{k,r} \end{cases}$$

$$\chi_{(k;r)}(n) = \begin{cases} 0, & n \notin Q_{(k;r)} \\ 1, & n \in Q_{(k;r)} \end{cases}$$

$a|b$ means a divides b

$[x]$ = the greatest integer $\leq x$

CHAPTER I

INTRODUCTION

For a fixed integer k, $k \geq 2$, recall that an integer n is said to be k-free if it is not divisible by the k-th power of any prime. There is a considerable literature concerning the k-free numbers. Let $q_k(x)$ be the number of k-free integers not exceeding x. Write

$$(1.1) \qquad q_k(x) = \frac{x}{\zeta(k)} + E(x) .$$

Then, Gagenbauer [10] proved in 1885 that

$$(1.2) \qquad E(x) = 0(x^{1/k}) , \quad \text{as} \quad x \to \infty$$

where the constant implied by the 0-estimate depends only on k. In 1931, Evelyn and Linfoot [8] improved this result to

$$(1.3) \quad E(x) = 0(x^{1/k} \exp \{-b(\log x \log \log x)^{1/2}\})$$

where b is an absolute constant and the constant implied by the 0-estimate depends only on k. They also showed that $E(x) \neq o(x^{1/2k})$. Axer [1], in 1911, showed that on the basis of the Riemann hypothesis,

$$(1.4) \qquad E(x) = 0(x^{(2/2k+1)+\epsilon})$$

where ϵ is any fixed positive number. Stark [16] remarked that if we assume $E(x) = 0(x^{(1/2k)+\epsilon})$, for any fixed $k \geq 2$ and all $\epsilon > 0$,

this implies the Riemann hypothesis. Very recently, in 1976, Montgomery and Vaughan announced that for $k = 2$, on the basis of the Riemann Hypothesis, $E(x) = O(x^{(21/62)+\epsilon})$ for any fixed $\epsilon > 0$. This is a considerable improvement over Axer's result. However, their improvement, which is as yet unpublished, is applicable to the case $k = 2$ only.

Let D_k and d_k denote the asymptotic and Schnirelmann densities, respectively, of the k-free integers, i.e.:

(1.5) $$D_k = \lim_{x \to \infty} q_k(x)/x = 1/\zeta(k)$$

(1.6) $$d_k = \inf_{n > 0} q_k(n)/n \quad .$$

Obviously, $d_k \leq D_k$. One is immediately led to ask if $d_k < D_k$ and, if $d_k < D_k$, how large is the difference between d_k and D_k and at what value of n is $q_k(n)/n = d_k$, (that such a value exists whenever $d_k < D_k$ follows trivially from (1.1) and (1.2)). We shall in the discussion that follows, denote by n_k any number n such that $d_k = q_k(n)/n$. The above questions were first answered for $k = 2$ by K. Rogers [14] who proved that $d_2 = 53/88$ and $n_2 = 176$. Stark [16], using analytic methods, proved that $d_k < D_k$ for all $k \geq 2$. Unfortunately, his proof yielded no information about the value of n_k or the size of the difference $D_k - d_k$. Orr [12] made an extremely important contribution to this topic when he produced an elementary proof for Stark's result, proving also that for $k \geq 5$, $5^k \leq n_k < 6^k$. Orr also published the values of n_k and d_k for $k = 3, 4, 5,$ and 6 (these values are listed in Table 1 in

Appendix A. Duncan [4] proved that the asymptotic and Schnirelmann densities interlace, i.e.:

(1.7) $$d_k < D_k < d_{k+1} < D_{k+1} \quad .$$

In a later paper, Duncan [5] also found that

(1.8) $$d_k > 1 - \sum_{p \text{ prime}} p^{-k}$$

and that d_{k+1} is closer to D_{k+1} than to D_k, in fact

(1.9) $$\frac{D_{k+1} - d_{k+1}}{D_{k+1} - d_k} < 2^{-k} \quad .$$

In a recent paper, Diananda and Subbarao [2] showed that:

(1.10) $$\frac{1}{2} \cdot 6^k \leq n_k < 6^k$$

and that either:

i) $3^k | n_k$ or $5^k | n_k$ or

ii) $2^k | n_k$ and either a multiple of 3^k or 5^k occurs between n_k and $n_k - 2^k$.

Also shown by Diananda and Subbarao was that

(1.11) $$d_k \geq 1-2^{-k}-3^{-k}-5^{-k}+(3^{-k}+2\cdot 5^{-k})/(6^k-3^k+1) \quad .$$

A major contribution of this dissertation shall be to extend the study of the Schnirelmann density of the k-free integers. The principal questions dealt with are:

1) What is the best estimate for the value of E_k which we define by:

$$E_k = d_k - (1-2^{-k}-3^{-k}-5^{-k}) .$$

2) What is the value of n_k (which in light of (1.10) we define to be the smallest n on $[\frac{1}{2} \cdot 6^k, 6^k)$ at which d_k is achieved)?

Notice that by (1.11), $E_k > 0$.

To answer these questions, we use a theoretical investigation in Chapter II and a computational investigation in which the values of n_k and d_k are calculated for $7 \leq k \leq 75$, in Chapter III.

The first result obtained is a simplified proof that $d_k < D_k$. Let us introduce some notation:

Let

$$E_k(n) = \frac{q_k(n)}{n} - \{1-2^{-k}-3^{-k}-5^{-k}\}$$

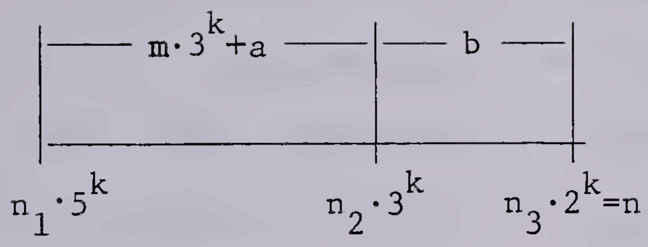
$$M_m(n) = n - ([\frac{n}{m}] \cdot m) .$$

We show that for $\frac{1}{2} \cdot 6^k \leq n < 6^k$,

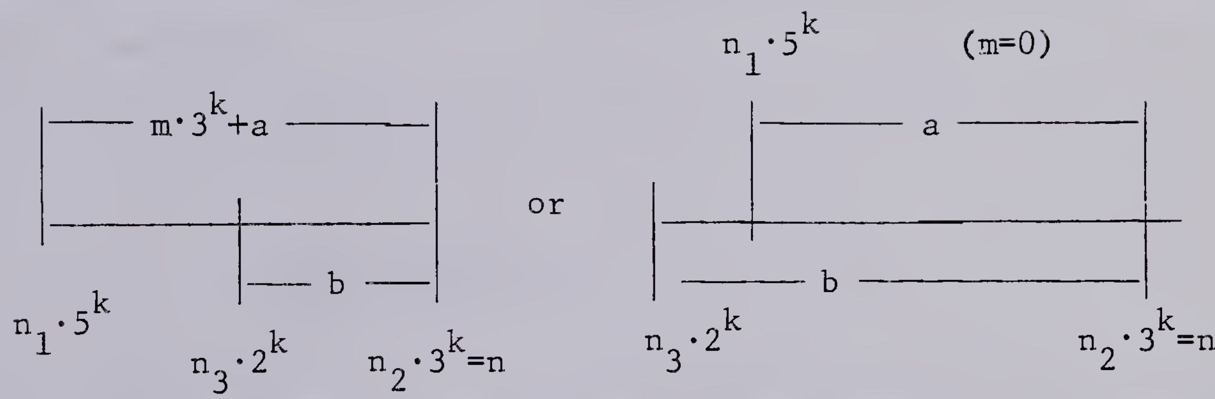
$$E_k(n) = \frac{1}{n}\{2^{-k} \cdot M_{2^k}(n) + 3^{-k} \cdot M_{3^k}(n) + 5^{-k} \cdot M_{5^k}(n)\} . \tag{1.12}$$

Using this result, we categorize the values of n which could be equal to n_k. We find that n must satisfy one of four cases shown in the following diagram:

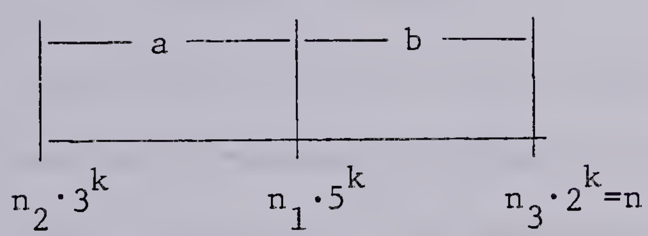
(i)



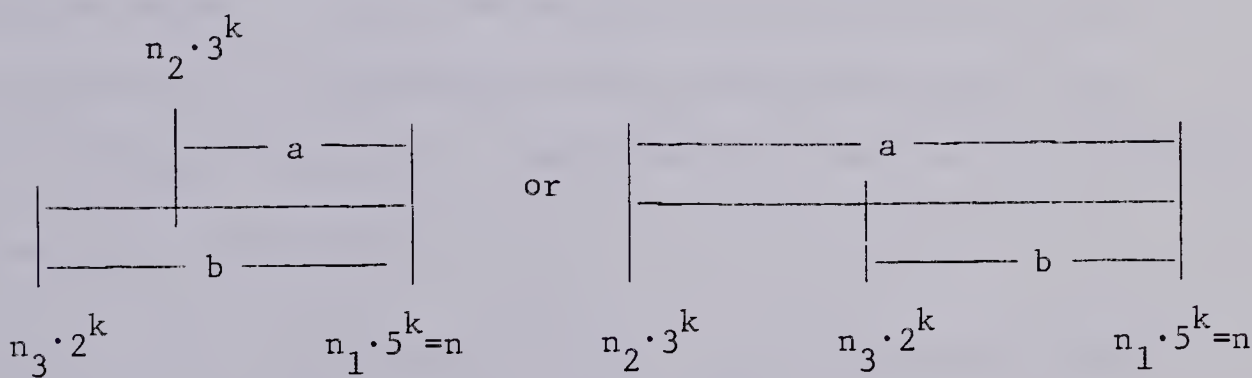
(ii)



(iii)



(iv)



where $0 < a < 3^k$, $0 < b < 2^k$, $0 \leq m < (\frac{5}{3})^k$, a, b and m are integers.

Applying (1.12) to case (i), we prove

$$d_k \leq 1-2^{-k}-3^{-k}-5^{-k}+\frac{([3^k+2^k-2]/5^k)+([2^k-1]/3^k)}{6^k-5^k+2^k+1} \quad . \tag{1.13}$$

We then prove that, in the above diagram, we can assert that $0 \leq m < 2 \cdot (10/9)^k$, and that there are at most $0(\frac{4}{3})^k$ values at which n_k can occur. This represents a substantial improvement over the result of Diananda and Subbarao, which gives an estimate $0(2^k)$ values at which n can occur.

We also show that

$$\frac{E_k}{D_k - d_k} = 0(\frac{2}{3})^k \to 0$$

as $k \to \infty$, which answers a query raised by Carl Pomerance who asked if $E_k/D_k - d_k$ converged to zero.

Using the new theoretical arguments from Chapter II, we were able to construct a computer program to calculate the values of n_k and d_k for $7 \leq k \leq 75$. This represents an enormous advance in the number of known values of d_k. The details of the calculations are given in Chapter III. On the basis of the calculated results, we formed the following conjectures, which describe very well the behavior of E_k and n_k, $5 \leq k \leq 75$:

Conjecture A: For n sufficiently large, n_k occurs at the first multiple of 2^k following the first multiple of 3^k following some multiple of 5^k on the interval $(\frac{1}{2} \cdot 6^k, 6^k)$.

Conjecture B: Let $G(k) = 1 - \exp\{-\frac{7}{48}(108)^k (E_k)^2\}$ and $g(k,x) =$ the number of $\hat{k}$, $2 \leq \hat{k} \leq k$ such that $G(\hat{k}) < x$. Then for any fixed x, $0 \leq x \leq 1$:

$$\lim_{k \to \infty} \frac{g(k,x)}{x} = x \quad .$$

Conjecture C: Let $h(k,x)$ = the number of $\hat{k}$, $5 \leq \hat{k} \leq k$ such that $(n_{\hat{k}}/6^{\hat{k}}) < x$. Then for any fixed x, $\frac{1}{2} \leq x \leq 1$,

$$\lim_{k \to \infty} \frac{h(k,x)}{k} = \frac{8}{7}(x^3 - \frac{1}{8}) \quad .$$

While we believe that the conjectures are likely true, we leave their proof as an open question.

A second major contribution of this dissertation is to unify the study of the k-free and the semi k-free integers. Recall that for $k \geq 2$, an integer n is said to be semi k-free if no k-th power of any prime unitarily divides n. (p^k unitarily divides n precisely when $p^k | n$ but $p \nmid \frac{n}{p^k}$.) In the literature it does not seem to have been noticed that the study of these two classes of integers can be unified. We do this by studying what we call the (k,r)-free integers. Let us introduce a definition:

Definition (1.14): Let $1 \leq r < k \leq \infty$. Then if n is any positive integer, n is said to be (k,r)-free if, in its canonical form $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i}$, each α satisfies either $\alpha < r$ or $\alpha \geq k$. Let $Q_{(k;r)}$ be the set of all (k,r)-free integers.

Remark: An equivalent definition for the (∞,r)-free integers is to define n to be (∞,r)-free if

$$n \in \bigcap_{k=r+1}^{\infty} Q_{(k;r)} \quad .$$

It is trivial that the r-free integers are a special case of

the (k,r)-free integers $(k = \infty,\ r \geq 2)$, and that the semi r-free integers are also a special case of the (k,r)-free intergers $(k = r+1)$. Another interesting case arises when $r = 1$ -- the so called k-full integers, but in our discussion we consider only the case $r \geq 2$. We obtain a number of asymptotic results for the (k,r)-free integers in which the error estimates are independent of the value of k, and thus unify the study of the r-free and semi r-free integers. Indeed, some of the results, in the special case of the semi r-free integers, have not previously been published.

Let $q_{(k;r)}(x)$ = the number of $n \leq x$ such that $n \in Q_{(k;r)}$. In Chapter IV, we find that for $2 \leq r < k \leq \infty$,

$$(1.15) \qquad q_{(k;r)}(x) = x \cdot \prod_p (1 - p^{-r} + p^{-k}) + E(x)$$

where

$$(1.16) \qquad E(x) = O(\frac{k}{k-r} \cdot x \cdot \exp(-C_r(\log^{3/5} x)(\log\log x)^{-1/5}))$$

as $x \to \infty$ where C_r depends only on r and the O-estimate is uniform in k and r.

If the Riemann hypothesis is true, then we find

$$(1.17) \qquad E(x) = O(\frac{k}{k-r} x^{1/r} w(x) \cdot x^{-(\frac{k-r}{r(2rk+k-2r)})}), \quad \text{as } x \to \infty$$

where

$$(1.18) \qquad w(x) = \exp\{A \log x\ (\log\log x)^{-1}\}$$

and A is an absolute positive constant and the O-estimate is

uniform in k and r.

In Chapter V we prove, by elementary means, that the asymptotic and Schnirelmann densities of the (k,r)-free integers are different (except possibly when $(k,r) = (2,3)$). The Schnirelmann densities for the (k,r)-free integers for all $2 \leq r \leq 20$ and all k (except for the case of the $(2,3)$-free integers) are listed.

In Chapter V, we also study the Schnirelmann density of the (k,r) integers. This class of integers, which have been studied by Feng and Subbarao [17] and others, is defined as follows:

Definition (1.19). Let n be a positive integer, and let $2 \leq r < k < \infty$. Then n can be uniquely represented as $n = d^k m$ where m is k-free. If m is also r-free, then n is said to be a (k,r) integer. Again, n is an (∞,r) integer if it is a (k,r) integer for all $k > r$.

The Schnirelmann density of the (k,r) integers in the special case $r = 2$, $k \geq 4$, has been studied by Diananda [3]. In Chapter V we prove, again by elementary means, that for all $2 \leq r < k \leq \infty$ the Schnirelmann and asymptotic densities of the (k,r) integers are different. Also, the Schnirelmann densities of the (k,r) integers for all $2 \leq r \leq 20$ and all k are calculated. It is further shown that for all $2 \leq r < k \leq \infty$, except the case $k = 3$, $r = 2$ and (possibly) when $\frac{15^r - 9^r}{5^r + 3^r} < 2^k < 3^r$, we have the Schnirelmann densities of the (k,r)-free integers and the (k,r) integers are the same and are achieved at the same point. We also characterize in detail the intervals on which these Schirelmann densities can be achieved.

These results can be further generalized. One possible generalization is to consider the following class of integers suggested by Professor L. Carlitz. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ such that $1 \leq \beta_i < \alpha_i \leq \infty$ and the squence of primes $2,3,5,7,\ldots = p_1, p_2, p_3, \ldots$ and any integer $n = p_{\gamma_1}^{\delta_1} \cdot p_{\gamma_2}^{\delta_2} \cdot \ldots \cdot p_{\gamma_i}^{\delta_i}$. Then we say n is (α,β)-free if $\delta_j < \beta_{\gamma_j}$ or $\delta_j \geq \alpha_{\gamma_j}$ for $1 \leq j \leq i$. It can easily be shown that if all but a finite number of $\beta_i \geq r$, then

$$q_{(\alpha,\beta)}(x) = x \cdot \prod_{p_i} \left(1 - \frac{1}{p_i^{\beta_i}} + \frac{1}{p_i^{\alpha_i}} \right) + O(x^{1/r}) \quad \text{as} \quad x \to \infty$$

where $q_{(\alpha,\beta)}(x)$ is the number of (α,β)-free integers $\leq x$. A discussion of further problems and generalizations is given in Chapter VI.

CHAPTER II

ON THE SCHNIRELMANN DENSITY OF THE k-FREE INTEGERS

(§1) Introduction

Let d_k and D_k denote respectively the Schnirelmann and asymptotic densities of the k-free numbers. So far, no simple formula is known for d_k. In 1964, Kenneth Rogers [14] showed that $d_2 = 53/88 < D_2 = 6/\pi^2$. In fact, $D_2 = q_2(176)/176$. ($q_r(n)$ is used to denote the number of positive r-free integers that are $\leq n$). In 1966, H.M. Stark [16] showed by analytic methods the general result that for all $k \geq 2$,

$$d_k < D_k. \tag{2.1.1}$$

It should be remarked that this implies that d_k is attained at some value of n, say n_k. It is clear that

$$Q_2 \subset Q_3 \subset Q_4 \quad ,$$

where Q_r denotes the set of all r-free integers, and hence that

$$d_2 \leq d_3 \leq d_4 \leq \dots \leq d_k \leq \dots \leq 1 \quad .$$

As R.L. Duncan [4] observed, it is easily shown that for $k \geq 2$,

$$d_k < D_k < d_{k+1} < D_{k+1} \quad .$$

In a later paper, Duncan [5] proved that

$$\frac{D_{k+1} - d_{k+1}}{D_{k+1} - d_k} < \frac{1}{2^k} \quad , \tag{2.1.2}$$

and thus d_{k+1} is closer to D_{k+1} than to D_k.

In 1969, Orr [12] gave a non-analytic proof of (2.1.1). He also proved that for $k \geq 5$, all n_k for a given k satisfy

(2.1.3) $$5^k \leq n_k < 6^k \quad .$$

Orr also calculated d_k and n_k for $3 \leq k \leq 6$: $d_3 = 157/189$, $d_4 = 145/157$, $d_5 = 3055/3168$, $d_6 = 6165/6272$, $n_3 = 378$, $n_4 = 2512$, $n_5 = 3168$ or 6336, and $n_6 = 31360$.

In a recent paper, P.H. Diananda and M.V. Subbarao [2] obtained, among other results, the following:

(2.1.4) $$d_k > 1-2^{-k}-3^{-k}-5^{-k} \quad .$$

For $k \geq 5$, the largest n_k for a fixed k satisfies:

(2.1.5) $$\frac{1}{2} \cdot 6^k \leq n_k < 6^k$$

and

Theorem (2.1.6): For $k \geq 5$, any n_k satisfying (2.1.5) also satisfies:

1) $3^k | n_k$ or $5^k | n_k$ or

2) $2^k | n_k$ and between $n_k - 2^k$ and n_k there is a multiple of 3^k or 5^k.

They also proved that for $k \geq 5$,

(2.1.7) $$d_k \geq 1-2^{-k}-3^{-k}-5^{-k}+(3^{-k}+2\cdot 5^{-k})(6^k-3^k+1)^{-1} \quad .$$

These results are the starting point for our investigation of d_k and n_k. In (§2), we begin our investigation by considering the functions E_k and $E_k(n)$ which we define as follows:

$$E_k = d_k - (1-2^{-k}-3^{-k}-5^{-k}) \tag{2.1.8}$$

$$E_k(n) = \frac{q_k(n)}{n} - (1-2^{-k}-3^{-k}-5^{-k}) \quad .$$

The principal result of (§2) will be to find the upper bound for d_k stated in (2.2.6). This is the best general upper bound for d_k which has yet been found. The upper bound is used to provide a completely elementary proof of (2.1.1).

In (§3) we consider the number of integers between $\frac{1}{2} \cdot 6^k$ and 6^k at which d_k can be attained. From Theorem (2.1.6), that number is $O(2^k)$ as $k \to \infty$. We improve this to $O((\frac{4}{3})^k)$. This improvement is of great value in divising a computer program to calculate n_k. (The details of this program, and computed values of d_k and n_k for $7 \leq k \leq 75$ are given in Chapter 3.)

In (§4), we propose three conjectures, conjectures (2.4.1), (2.4.2), and (2.4.3) which predict the behavior of n_k and d_k as $k \to \infty$. We show, to support these conjectures, that the actual values of d_k and n_k as calculated in Chapter 3 are predicted very well by these conjectures.

(§2) An Upper Bound for E_k

Before proceeding to the main results, we introduce some notations:

$$(2.2.1) \qquad R_m(n) = [n/m] \cdot m$$

$$M_m(n) = n - R_m(n) \quad .$$

Thus $R_m(n)$ is the greatest multiple of m which is less than or equal to n, and $M_m(n) \equiv n(\text{mod } m)$. We shall now prove a basic lemma:

Lemma (2.2.2): For $k \geq 2$, n a positive integer, if $1 \leq n < 6^k$, we have:

$$(2.2.3) \qquad E_k(n) = (2^{-k} M_{2^k}(n)+3^{-k} M_{3^k}(n)+5^{-k} M_{5^k}(n))n^{-1} \quad ,$$

where $E_k(n)$ is defined by Equation (2.1.8).

Proof: Since $n < 6^k$,

$$q_k(n) = n - [n \cdot 2^{-k}] - [n \cdot 3^{-k}] - [n \cdot 5^{-k}] \quad .$$

Thus we have:

$$q_k(n)/n = 1 - ([n \cdot 2^{-k}] + [n \cdot 3^{-k}] + [n \cdot 5^{-k}])n^{-1}$$

$$= 1 - \left((n-M_{2^k}(n)) \cdot 2^{-k} + (n-M_{3^k}(n)) \cdot 3^{-k} + (n-M_{5^k}(n)) \cdot 5^{-k}\right) n^{-1} \quad .$$

Combining (2.1.8) and the above result, we easily obtain (2.2.3).

Q.E.D.

Remark: In view of (2.1.5) and since

$$q_k(n)/n = (1-2^{-k}-3^{-k}-5^{-k}) + E_k(n) \quad ,$$

to find n_k and d_k, we must minimize $E_k(n)$ for $\frac{1}{2} \cdot 6^k \leq n < 6^k$.

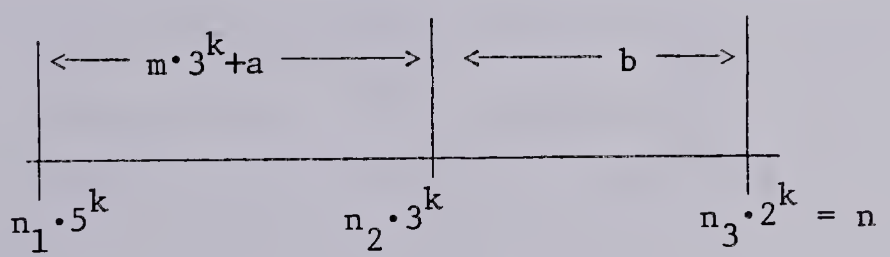
But by Lemma (2.2.2), $E_k(n+1) > E_k(n)$ except when $n+1$ is a multiple of 2^k, 3^k or 5^k. Also, if no multiple of 3^k or 5^k lies between n and $n+2^k$, $E_k(n+2^k) > E_k(n)$. Thus Theorem (2.1.6) follows from Lemma (2.2.2) and hence Lemma (2.2.2) provides an alternate proof of Theorem (2.1.6).

Using Theorem (2.1.6), we categorize all n in the interval $[\frac{1}{2} \cdot 6^k, 6^k)$ at which d_k could be attained, into four categories, given in Diagram (2.2.4). We assume this n satisfies $n_1 5^k \leq n <$ $< (n_1+1)\, 5^k$, where n_1 is an integer such that $(n_1+1)\cdot 5^k > \frac{1}{2} \cdot 6^k$ but $n_1 \cdot 5^k < 6^k$.

Diagram (2.2.4)

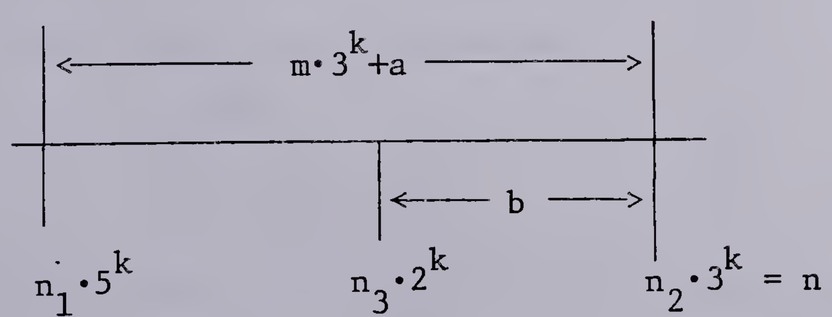
<u>Four Possible Types of Numbers Which Can be</u> n_k

(2.2.4.1)



, (Type 1)

where $n_2 \cdot 3^k$ is the multiple of 3^k such that m other multiples of 3^k occur between $n_1 \cdot 5^k$ and $n_2 \cdot 3^k$, a is the difference between $n_1 \cdot 5^k$ and the first multiple of 3^k greater than $n_1 \cdot 5^k$, $n_3 \cdot 2^k = n$ is the first multiple of 2^k greater than $n_2 \cdot 3^k$ and $b = (n_3 \cdot 2^k) - (n_2 \cdot 3^k)$.

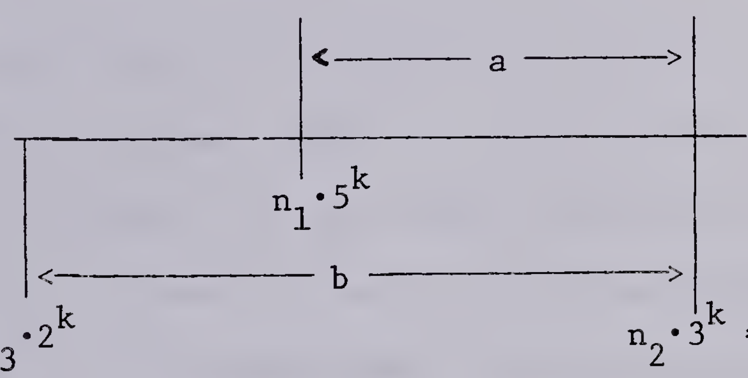


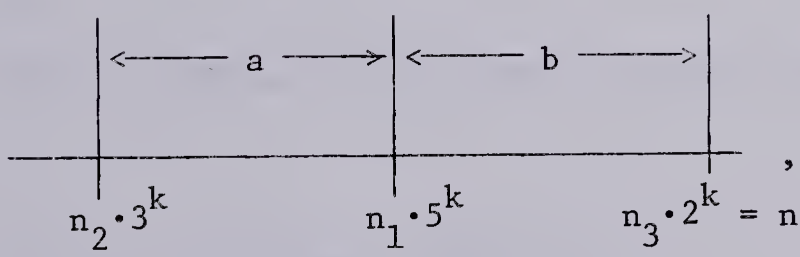
, (Type 2)

(2.2.4.2)

where $n_2 \cdot 3^k$, m and a are defined as for (2.2.4.1), $n_3 \cdot 2^k$ is the greatest multiple of 2^k less than $n_2 \cdot 3^k = n$, and $b = (n_2 \cdot 3^k) - (n_3 \cdot 2^k)$.

<u>Remark</u>: If $b > m\ 3^k + a$ (which implies $m = 0$), then $n_3 \cdot 2^k < n_1 \cdot 5^k$. This possibility, illustrated below, is still considered to be Case (2.2.4.2):





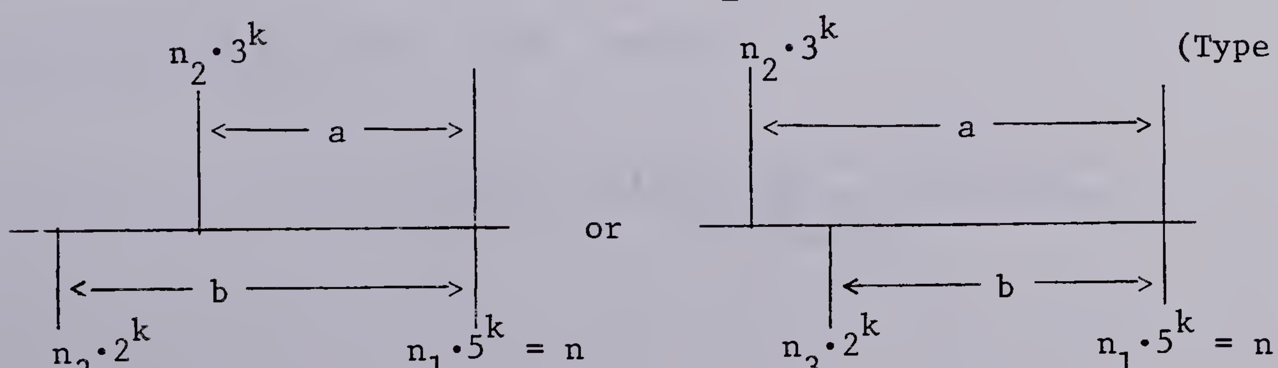
, (Type 3)

(2.2.4.3)

where $n_2 \cdot 3^k$ is the greatest multiple of 3^k less than $n_1 \cdot 5^k$, $n_3 \cdot 2^k$ is the smallest multiple of 2^k greater than $n_1 \cdot 5^k$, $a = (n_1 \cdot 5^k) - (n_2 \cdot 3^k)$ and $b = (n_3 \cdot 2^k - n_1 \cdot 5^k)$.

<u>Remark</u>: If there exists a multiple of 3^k between $n_3 \cdot 2^k = n$ and $n_1 \cdot 5^k$, we consider n to be of type (2.2.4.1) and say there is no n of type (2.2.4.3) corresponding to $n_1 \cdot 5^k$.

(Type 4)



,

(2.2.4.4)

where $n_2 \cdot 2^k$ and a are defined as for (2.2.4.3), $n_3 \cdot 2^k$ is the greatest multiple of 2^k less than $n_1 \cdot 5^k = n$, and $b = (n_1 \cdot 5^k) + -(n_3 \cdot 2^k)$. The two diagrams illustrate the cases $a < b$ and $a > b$.

Claim: In (2.2.4.1) and (2.2.4.2), $0 \leq m < (5/3)^k$ while in (2.2.4.1) through (2.2.4.4), $0 < a < 3^k$ and $0 < b < 2^k$.

Proof: We stated above that $n < (n_1+1)\, 5^k$, but from (2.2.4.1) or (2.2.4.2), $n > 3^k m + 5^k n_1$. Thus $m < (5/3)^k$. Since we always take $n < 6^k$, all multiples of 2^k, 3^k and 5^k in the above diagrams are distinct, and thus $0 < a$ and $0 < b$. Since a and b are always defined in terms of a multiple of 3^k or 2^k respectively, it easily follows $a < 3^k$ and $b < 2^k$. Q.E.D.

We use Lemma (2.2.2) in each of these four cases to find $E_k(n)$:

(2.2.5)
$$
\begin{aligned}
E_k(n) &= ((m \cdot 3^k + a + b)5^{-k} + b \cdot 3^{-k})n^{-1}, && \text{Case (2.2.4.1)}\\
&= ((m \cdot 3^k + a)5^{-k} + b \cdot 2^{-k})n^{-1}, && \text{Case (2.2.4.2)}\\
&= ((a+b)3^{-k} + b \cdot 5^{-k})n^{-1}, && \text{Case (2.2.4.3)}\\
&= (a \cdot 3^{-k} + b \cdot 2^{-k})n^{-1}, && \text{Case (2.2.4.4)}
\end{aligned}
$$

We shall use (2.2.5), Case (2.2.4.1) to find an upper bound for d_k:

Theorem (2.2.6): For $k \geq 2$, we have:

(2.2.7)
$$d_k \leq 1 - 2^{-k} - 3^{-k} - 5^{-k} + \frac{((3^k + 2^k - 2)5^{-k} + (2^k - 1)3^{-k})}{6^k - 5^k + 2^k + 1} .$$

Proof: Let $k \geq 7$. Let $\hat{n}$ be the largest n on $[\frac{1}{2} \cdot 6^k, 6^k)$ satisfying Case (2.2.4.1) such that $m = 0$. We propose to find an upper bound for $E_k(\hat{n})$. (We take this case since it will produce the sharpest upper bound for $E_k(\hat{n})$.)

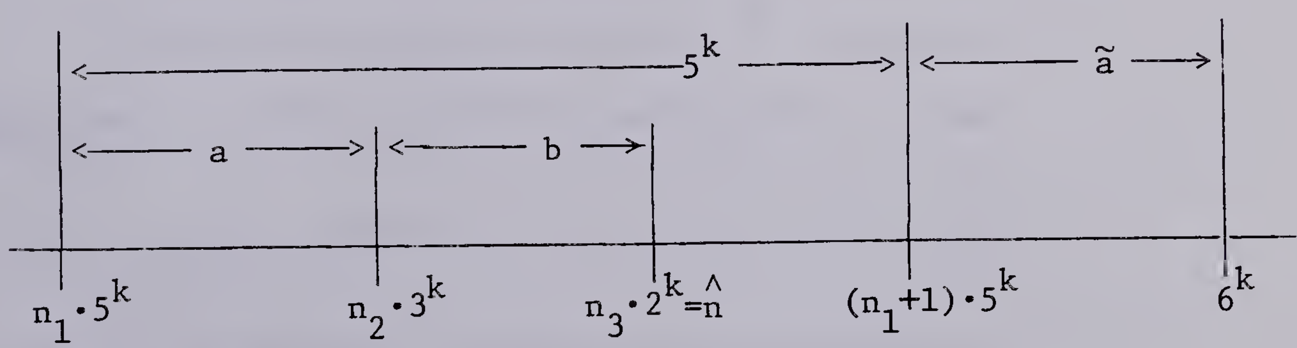
We have, from (2.2.5)

$$E_k(\hat{n}) = ((a+b)5^{-k} + b \cdot 3^{-k})\hat{n}^{-1} \quad , \tag{2.2.8}$$

where a and b depend on $\hat{n}$ as described in Diagram (2.2.4.1). Also, $0 < a < 3^k$ and $0 < b < 2^k$.

Let us initially assume $a = 3^k - 1$ and $b = 2^k - 1$. It is clear that either $\hat{n}$ is greater than the largest multiple of 5^k which is less than 6^k (Case A), or $\hat{n}$ is not greater than this multiple but is greater than the second largest multiple of 5^k which is less than 6^k (Case B). Let us assume Case B holds, as illustrated by Diagram (2.2.9), where $(n_1+1) \cdot 5^k$ is the largest multiple of 5^k which is less than 6^k, and $n_1 \cdot 5^k$, $n_2 \cdot 3^k$, $n_3 \cdot 2^k = \hat{n}$, a and b are defined as in Diagram (2.2.4.1), and $\tilde{a} = 6^k - (n+1) \cdot 5^k$.

Diagram (2.2.9)



We have assumed $\hat{n} < (n_1+1) \cdot 5^k$. However, if $(n_1+1) \cdot 5^k < 6^k - 3^k$, we must have a multiple of 3^k lying strictly between $(n_1+1) \cdot 5^k$ and 6^k. Call this first such multiple $\tilde{n}_2 \cdot 3^k$.

But then the first multiple of 2^k greater than $\tilde{n}_2 \cdot 3^k$ is less than 6^k (since $\tilde{n}_2 \cdot 3^k \le 6^k - 3^k < 6^k - 2^k$) and this multiple would be $\hat{n}$, and we would have Case A.

Thus, in Case B, we must assume $(n_1+1) \cdot 5^k > 6^k - 3^k$, i.e., $\tilde{a} \le 3^k - 1$. Clearly

$$\hat{n} = 6^k - \tilde{a} - 5^k + a + b \quad .$$

But, as assumed above, $a = 3^k - 1$ and $b = 2^k - 1$. Thus, taking into consideration $\tilde{a} \le 3^k - 1$,

$$\hat{n} \ge 6^k - 5^k + 2^k - 1 \quad .$$

Let us assume $\hat{n} = 6^k - 5^k + 2^k - 1$. But then $\tilde{a} = 3^k - 1$. But then we have $3^k | (6^k - 3^k) = ((n_1+1)\ 5^k - 1)$. Since $a = 3^k - 1$, $3^k | ((n_2 - 1)\ 3^k) = (n_1 \cdot 5^k - 1)$. Thus $3^k | 5^k$, a contradiction. Thus

$$\hat{n} \ge 6^k - 5^k + 2^k \quad .$$

However, $\hat{n}$, a multiple of 2^k, is even, while $(6^k - 5^k + 2^k)$ is odd. Thus we have:

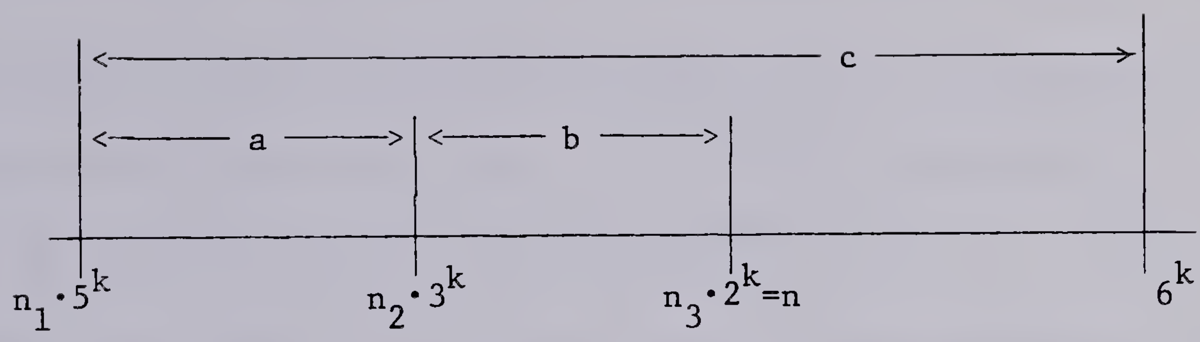
$$\hat{n} \ge 6^k - 5^k + 2^k + 1 \quad .$$

Substituting this estimate for $\hat{n}$, along with our assumptions $a = 3^k - 1$, $b = 2^k - 1$ into (2.2.8), we find

$$\text{(2.2.10)} \qquad E_k(\hat{n}) \le ((3^k + 2^k - 2)5^{-k} + (2^k - 1)3^{-k})(6^k - 5^k + 2^k + 1)^{-1} \quad .$$

Let us now assume Case A, as illustrated by Diagram (2.2.11).

Diagram (2.2.11)



Here $n_2 \cdot 3^k$, $n_3 \cdot 2^k$, a and b are defined as in Diagram (2.2.9) but $n_1 \cdot 5^k$ is the largest multiple of 5^k less than 6^k. Thus $c < 5^k$. Clearly,

$$\hat{n} = 6^k - c + a + b \quad .$$

Since $a = 3^k - 1$, $b = 2^k - 1$ and $c < 5^k$, we have

$$\hat{n} > 6^k - 5^k + 3^k + 2^k - 2$$

$$> 6^k - 5^k + 2^k + 1 \quad .$$

Thus (2.2.10) remains valid for Case A. An examination of the above arguments will show that an assumption that $a < 3^k - 1$ or $b < 2^k - 1$ would decrease the upper bound for $\hat{n}$. However, we have shown above that

$$\hat{n} \geq 6^k - 3^k - 5^k + a + b + 1 \quad . \tag{2.2.12}$$

Let us assume $a = \hat{a}$ and $b = \hat{b}$ where $0 < \hat{a} \leq 3^k - 1$, $0 < \hat{b} \leq 2^k - 1$ and at least one of $\hat{a} \leq 3^k - 2$ or $\hat{b} \leq 2^k - 2$. Then, from (2.2.8) and (2.2.12) we have:

$$E_k(\hat{n}) \leq ((\hat{a}+\hat{b})5^{-k} + \hat{b} \cdot 3^{-k})(6^k - 3^k - 5^k + \hat{a} + \hat{b} + 1)^{-1} \quad . \tag{2.2.13}$$

Since $k \geq 7$, we can apply Lemma (2.2.20) to (2.2.12) to conclude

$$E_k(\hat{n}) < ((3^k+2^k-2)5^{-k}+(2^k-1)3^{-k})(6^k-5^k+2^k+1)^{-1}$$

and thus (2.2.10) is satisfied for any a or b. The theorem follows from (2.1.8) and (2.2.10), for $k \geq 7$. A direct calculation of E_2, E_3, E_4, E_5 and E_6 proves the theorem for $2 \leq k \leq 6$. Q.E.D.

Remark: The result, (2.2.7) proved here is an improvement over the result of Orr [12] that for $k \geq 5$,

$$(2.2.14) \qquad d_k < 1-2^{-k}-3^{-k}-5^{-k}+((\tfrac{2}{3})^k+(\tfrac{2}{5})^k+(\tfrac{3}{5})^k)(6^k-5^k-3^k-2^k)^{-1} \quad .$$

Corollary (2.2.15): For all $k \geq 2$, $d_k < D_k$.

Proof: As we remarked in Chapter 1, $D_k = 1/\zeta(k) = \prod_p (1-p^{-k})$, where $\zeta(k)$ is the well known Riemann zeta function. Let us take $k \geq 7$. We have

$$\begin{aligned} \zeta(k) &= \sum_{n=1}^{\infty} n^{-k} \\ (2.2.16) \qquad &< 1+2^{-k}+3^{-k}+4^{-k}+5^{-k}+6^{-k}+7^{-k}+8^{-k} + \int_8^{\infty} x^{-k}dx \\ &\equiv F(k) \quad , \end{aligned}$$

where $F(k)$ is defined by the right hand side of inequality (2.2.16). Thus we have

$$(F(k))^{-1} = (1+2^{-k}+\ldots+7^{-k}+(1+\frac{8}{k-1})8^{-k})^{-1} < D_k \quad .$$

In view of the above and (2.2.7), to prove the corollary, it is sufficient to prove that $G(k) < (F(k))^{-1}$ where $G(k)$ is defined to be the right hand side of (2.2.7). We have:

$$(F(k))^{-1}-G(k) = 840^k/(840^k+420^k+280^k+210^k+168^k+140^k+120^k+(1+\frac{8}{k-1})105^k)$$

$$+ (-1+2^{-k}+3^{-k}+5^{-k}) - \frac{((3^k+2^k-2)5^{-k}+(2^k-1)3^{-k})}{6^k-5^k+2^k+1} \quad .$$

Combining the first two terms, and eliminating some positive values, we obtain

$$(F(k))^{-1}-G(k) > \frac{4200^k-3600^k-(\frac{8}{k-1})3150^k}{30^k(840^k+420^k+\ldots+120^k+(1+\frac{8}{k-1})105^k} + \tag{2.2.17}$$

$$- \frac{6^k+9^k+10^k}{(6^k-5^k+2^k+1)15^k} \quad .$$

But for $k \geq 7$, we have:

$$4200^k-3600^k-(\frac{8}{k-1})3150^k > \frac{1}{3}\cdot 4200^k$$

$$840^k+420^k+280^k+210^k+168^k+140^k+120^k+(1+\frac{8}{k-1})105^k < \frac{3}{2}\cdot 840^k$$

$$6^k+9^k+10^k/(6^k-5^k+2^k+1)15^k < 3\cdot 9^{-k} \quad .$$

Substituting the above into (2.2.17), we have:

$$\begin{aligned}(F(k))^{-1}-G(k) &\geq \frac{2}{9}\cdot 4200^k\cdot(30\cdot 840)^{-k}-3\cdot 9^{-k} \\ &= \frac{2}{9}\cdot 6^{-k}-3\cdot 9^{-k} \\ &> 0 \quad .\end{aligned}$$

Thus the corollary is true for $k \geq 7$. For $2 \leq k \leq 6$, we observe that $q_k(n)/n < D_k$ for the following values of n:

k :	2	3	4	5	6
n :	176	378	2512	3168	31360 .

Thus the corollary is true for all $k \geq 2$. Q.E.D.

Remark: This proof that $d_k < D_k$ is completely elementary.

Corollary (2.2.18):

$$\frac{E_k}{D_k - d_k} = \frac{d_k - (1-2^{-k}-3^{-k}-5^{-k})}{D_k - d_k} = 0(\tfrac{2}{3})^k \quad \text{as} \quad k \to \infty . \tag{2.2.19}$$

Proof:

$$D_k = \frac{1}{\zeta(k)} = \prod_p (1-p^{-k})$$

$$= 1-2^{-k}-3^{-k}-5^{-k}+6^{-k}+0(7^{-k}) \quad .$$

By (2.2.7)

$$d_k-(1-2^{-k}-3^{-k}-5^{-k}) = 0(9^{-k}) \quad .$$

Thus,

$$\frac{E_k}{D_k-d_k} = \frac{0(9^{-k})}{6^{-k}+0(7^{-k})+0(9^{-k})}$$

$$= 0((\tfrac{2}{3})^k) \quad .$$ Q.E.D.

Remark: This corollary answers a query raised by Carl Pomerance [13] who asked if $(E_k/D_k-d_k) \to 0$ as $k \to \infty$.

We conclude this section by proving the lemma used in the proof of Theorem (2.2.6).

Lemma (2.2.20): If $k \geq 7$, and if $0 \leq a \leq 3^k-1$ and $0 \leq b \leq 2^k-1$, and at least one of $a < 3^k-1$ or $b < 2^k-1$, then

$$(2.2.21) \qquad ((a+b)5^{-k}+b\cdot 3^{-k})(6^k-5^k-3^k+a+b+1)^{-1} <$$

$$< ((3^k+2^k-2)5^{-k}+(2^k-1)3^{-k})(6^k-5^k+2^k+1)^{-1} .$$

<u>Proof</u>: Let us define a function $\tilde{E} = \tilde{E}(\tilde{a},\tilde{b},n)$ by:

$$\tilde{E}(\tilde{a},\tilde{b},n) = ((\tilde{a}+\tilde{b})5^{-k}+\tilde{b}\cdot 3^{-k})n^{-1} .$$

Let

$$E_1 = \tilde{E}(a,b,(6^k-5^k-3^k+a+b+1)) ,$$

$$E_2 = \tilde{E}((3^k-1),(2^k-1),(6^k-5^k-3^k+a+b+1) ,$$

$$E_3 = \tilde{E}((3^k-1),(2^k-1),(6^k-5^k+2^k+1)) ,$$

$$c = ((3^k-1)-a)+((2^k-1)-b) ,$$

$$d = (6^k-5^k+2^k+1)-(6^k-5^k-3^k+a+b+1) .$$

We wish to show $E_1 < E_3$. A simple calculation will show that $d = c+2$. We also know, by the hypothesis of the lemma, that $c \geq 1$. We also know (since $k \geq 7$) that $\tilde{E}$ is evaluated at n strictly on the interval $(\frac{1}{2}\cdot 6^k, 6^k)$. Let us consider

$$(2.2.22) \qquad \frac{\partial\tilde{E}}{\partial\tilde{a}} = 5^{-k}\cdot n^{-1} > 30^{-k} ,$$

since $n < 6^k$

$$(2.2.23) \qquad \frac{\partial\tilde{E}}{\partial\tilde{b}} = (5^{-k}+3^{-k})n^{-1} > 30^{-k} ,$$

since $n < 6^k$

$$\frac{\partial\tilde{E}}{\partial n} = -((\tilde{a}+\tilde{b})5^{-k}+\tilde{b}\cdot 3^{-k})n^{-2} .$$

Since $n > \frac{1}{2}\cdot 6^k$, $\tilde{a} < 3^k$, and $\tilde{b} < 2^k$, we have

$$(2.2.24) \qquad -\frac{\partial\tilde{E}}{\partial n} = ((\tilde{a}+\tilde{b})5^{-k}+\tilde{b}\cdot 3^{-k})n^{-2}$$

$$< 4((3^{k}+2^{k})5^{-k}+2^{k}\cdot 3^{-k})36^{-k}$$

$$= 4(54^{-k}+60^{-k}+90^{-k}) \quad .$$

In view of (2.2.22), (2.2.23) and the mean value theorem, we have:

$$E_2-E_1 > c\cdot 30^{-k} \quad .$$

By (2.2.24) and the mean value theorem, we have

$$E_2-E_3 = -\,d\cdot \frac{\partial\tilde{E}}{\partial n}\bigg|_{n=n_o} \quad ,$$

for n_o satisfying $(6^k-5^k-3^k+a+b+1) < n_o < (6^k-5^k+2^k+1)$. Thus,

$$E_2-E_3 < d\cdot 4(54^{-k}+60^{-k}+90^{-k})$$

$$= (c+2)\cdot 4(54^{-k}+60^{-k}+90^{-k}) \quad .$$

Thus we have:

$$E_3-E_1 = (E_2-E_1)-(E_2-E_3)$$

$$> c\cdot 30^{-k}-(4c+8)(54^{-k}+60^{-k}+90^{-k}) \quad .$$

Since $k \geq 7$, $30^{-k} > 12(54^{-k}+60^{-k}+90^{-k})$. Thus, since $c \geq 1$, we have:

$$E_3-E_1 > 30^{-k}-12(54^{-k}+60^{-k}+90^{-k}) > 0 \quad ,$$

this proves our result. Q.E.D.

(§ 3) The Number of Possible Values for n_k

In this section, we consider the number of possible values at which d_k can be achieved. As remarked earlier, Theorem (2.1.6) restricts the number of such n to $O(2^k)$ different values (as $k \to \infty$). Using (2.2.5) and Theorem (2.2.6), we shall obtain the following improvement:

Theorem (2.3.1): For $k \geq 5$, d_k can be achieved at most at $O((\frac{4}{3})^k)$ possible values on the interval $[\frac{1}{2}\cdot 6^k, 6^k)$.

Proof: From Theorem (2.2.6), we have $E_k = O(9^{-k})$ as $k \to \infty$. By a simple calculation, we find for $k \geq 8$,

$$E_k < 2\cdot 9^{-k} \quad . \tag{2.3.2}$$

We verify (2.3.2) directly for $5 \leq k \leq 7$. Thus (2.3.2) holds for all $k \geq 5$. From (2.2.5), Cases (2.2.4.1) and (2.2.4.2), we have

$$E_k(n) > m(\frac{3}{5})^k n^{-1} \quad ,$$

and since $n < 6^k$,

$$E_k(n) > m\cdot 10^{-k} \quad . \tag{2.3.3}$$

Combining (2.3.2) and (2.3.3), we find

$$2\cdot 9^{-k} > m\cdot 10^{-k} \quad ,$$

thus

$$m < 2\cdot(\frac{10}{9})^k \quad . \tag{2.3.4}$$

This is a vast improvement over the previous obvious estimate that $m < (\frac{5}{3})^k$. Thus for each multiple of 5^k on the interval $[\frac{1}{2}\cdot 6^k, 6^k)$, we must check Case (2.2.4.1) and Case (2.2.4.2)

each for $(\tilde{m}+1)$ possible values at which d_k could be achieved, where $\tilde{m} = [2(\frac{10}{9})^k]$. Cases (2.2.4.3) and (2.2.4.4) each must be checked for one value. Thus for each multiple of 5^k, there are at most $(4\ (\frac{10}{9})^k + 6)$ values to check. Since there are at most $[\frac{1}{2}\cdot(\frac{6}{5})^k]+1$ multiplies 5^k on $[\frac{1}{2}\cdot 6^k, 6^k)$, there are at most

$$(4\ (\frac{10}{9})^k + 6)([\frac{1}{2}\cdot(\frac{6}{5})^k]+1) = 0((\frac{4}{3})^k)$$

values at which d_k could be achieved. Q.E.D.

<u>Remark (2.3.5)</u>: The idea of the proof of this result is of enormous value for calculating n_k. Following the idea of estimating the maximal value of m from E_k, $E_k(n)$ was examined for all values of n satisfying Cases (2.2.4.1)-(2.2.4.4) from Diagram (2.2.4) with the additional assumption that $m = 0$. The minimal $E_k(n)$ obtained for every k, $5 \leq k \leq 75$, allowed us to restrict our search to $0 \leq m \leq 2$, while for $k \geq 14$, we could restrict our search $m = 0$ or 1, and for all $k \geq 38$, an $E_k(n)$ was found so small that we could assume $m = 0$. Thus a value for d_k could be determined after examining $0((\frac{6}{5})^k)$ points. We illustrate the power of this technique by comparing the number of points we must compare $E_k(n)$ using it, for $k = 10$ and $k = 75$, with the number of points allowed by Theorem (2.1.6). For $k = 10$, $E_{10} = 1.58 \times 10^{-11}$. In view of (2.3.3), we have $m < .158$. Thus $m = 0$. There are 3 multiples of 5^k on $[\frac{1}{2}\cdot 6^k, 6^k)$, and thus there are 12 points to check. By Theorem (2.1.6), there are 1030 points to check. For $k = 75$, E_k is so small that $m = 0$, and there are 1,736,296 points to check. By Theorem (2.1.6), there are 3.78×10^{22} points to check. Thus the above idea is essential in the calcualtion of d_k for large k.

(§4) Three Conjectures About n_k and d_k

We now present three conjectures based on the behaviour of n_k and d_k as calculated for $5 \leq k \leq 75$.

Conjecture (2.4.1): For k sufficiently large, n_k is achieved at the first multiple of 2^k following the first multiple of 3^k following some multiple of 5^k on the interval $[\frac{1}{2}\cdot 6^k, 6^k)$.

Conjecture (2.4.2): Let $G(k)$ be defined by:

$$G(k) = 1 - \exp\left(-\frac{7}{46}(108)^k E_k^2\right) \tag{2.4.3}$$

and let $g(k,t)$ be the number of $\hat{k}$, $2 \leq \hat{k} \leq k$ such that $G(\hat{k}) < t$. Then for $0 \leq t \leq 1$,

$$\lim_{k \to \infty} \frac{g(k,t)}{k} = t \quad . \tag{2.4.4}$$

Conjecture (2.4.5): If we consider n'_k to be the smallest integer on $[\frac{1}{2}\, 6^k, 6^k)$ at which d_k is achieved, and $h(k,x)$ is defined to be the number of $\hat{k}$, $5 \leq \hat{k} < k$, such that $(\frac{n'_k}{6^k}) < x$, then if $\frac{1}{2} \leq x \leq 1$, we have:

$$\lim_{k \to \infty} \frac{h(k,x)}{k} = \frac{8}{7}\left(x^3 - \frac{1}{8}\right) \quad . \tag{2.4.6}$$

We are able to test the conjectures for $5 \leq k \leq 75$. We obtain the following results:

1) n'_k, the smallest number on $[\frac{1}{2}\cdot 6^k, 6^k)$ at which d_k is achieved, is a number of the type described by Conjecture (2.4.1)

for all k, $13 \leq k \leq 75$.

2) We plot a histogram of the number of $\hat{k}$, $6 \leq \hat{k} \leq 75$ for which $(n'_{\hat{k}}/6^k) < x$ verses a graph of $70(\frac{8}{7}(x^3 - \frac{1}{8}))$, the curve predicted by Conjecture (2.4.2). A close fit is obtained (cf., Graph 2).

3) We plot the number of $\hat{k}$, $5 \leq \hat{k} \leq 75$, such that $(1 - \exp(-\frac{7}{48}(108)^{\hat{k}}(E_{\hat{k}})^2) < x$ for $x = 0, \frac{1}{10}, \ldots, 1$. A very good linear fit is obtained, and thus the data is well predicted by Conjecture (2.4.5). However, the best fitting line does not pass through the origin. There are an excess of $\hat{k}$ such that $(1 - \exp(-\frac{7}{48}(108)^{\hat{k}}(E_{\hat{k}})^2)$ is near zero. We believe this excess arises from the fact that E_k is tested for small k (cf., Graph 1). Despite this, from the data obtained, we cannot exclude the possibility that the conjecture needs to be modified for small x.

Remark: Some of the results in this chapter have been published by the author with P. Erdös and M.V. Subbarao [7].

CHAPTER III

NUMERICAL RESULTS ON THE SCHNIRELMANN DENSITY OF THE k-FREE INTEGERS

(§1) An Algorithm for Determining d_k and n_k.

In this section, we describe our algorithm for calculating n_k and d_k, by which we extended the known values of d_k and n_k from $k = 7$ to $k = 75$. Based on this algorithm, we wrote a computer program in Fortran IV and employed the Amdahl computer installation at the University of Alberta to perform our calculations. (For precise details about the computer program, the reader may consult the author.) We shall now describe our algorithm.

Recall (2.1.5) that $\frac{1}{2} \cdot 6^k \leq n_k < 6^k$, and n_k is the point at which $E_k(n)$ (defined in (2.1.8)) is minimized. Let us define $F_k(n)$ by:

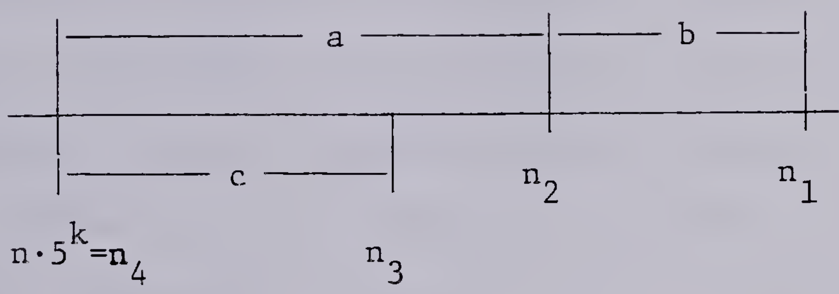
$$F_k(n) = n \cdot 30^k \cdot E_k(n) \quad . \tag{3.1.1}$$

Clearly, to find n_k, it is sufficient to minimize $F_k(n)/n$ on the interval $[\frac{1}{2} \cdot 6^k, 6^k)$. By Theorem (2.1.6), we know that n_k must be of the form shown in Diagrams (2.3.4.1) - (2.3.4.4). We initially will consider only a subset of the n of this form, the n in Diagrams (2.3.4.1) - (2.3.4.4) for which $m = 0$. We minimize $E_k(n)/n$ for these n, employing the following procedure.

For each multiple of 5^k on $[\frac{1}{2} \cdot 6^k, 6^k)$ (say $n \cdot 5^k$) there are not more than four numbers of the form shown in Diagram (2.3.4)

(with $m = 0$) on the interval $[n\cdot 5^k,(n+1)\cdot 5^k)$. These are n_1, n_2, n_3 and n_4 as diagrammed below.

(3.1.2)



where $n_3 = [\frac{n_4}{2^k}]\cdot 2^k+2^k$, $n_2 = [\frac{n_4}{3^k}]\cdot 3^k+3^k$, $n_1 = [\frac{n_2}{2^k}]\cdot 2^k+2^k$, $c = n_3-n_4$, $a = n_2-n_4$ and $b = n_1-n_2$. We start with $n\cdot 5^k = [\frac{6^k}{5^k}]\cdot 5^k$, and test n_1, then n_2, then n_3 and finally n_4 (as detailed below). We reduce n_4 by 5^k, calculate the new a , b and c and test the new n_1, n_2, n_3 and n_4. In this manner, we consider all the points in descending order.

<u>Remarks</u>: 1) If for $n_4 = [\frac{6^k}{5^k}]\cdot 5^k$, $n_2 = 6^k$, (which happens if and only if $n_3 = 6^k$ and $n_1 = 6^k+2^k$), we ignore n_1, n_2 and n_3 in our search. If for $n_4 = [\frac{(1/2)\cdot 6^k}{5^k}]\cdot 5^k$, any n_1, n_2 or n_3 lie in the interval $[\frac{1}{2}\cdot 6^k, 6^k)$, these n_1, n_2 and n_3 are included in our search).

2) It may happen (rarely) that $n_3 > n_2$. But then $n_3 = n_1$ and ignoring the second testing of n_1, we may still assert the n are tested in descending order.

3) To recalculate a, b and c requires modulo arithmetic on the old a, b and c. This involves only addition and subtraction,

which are very fast operations in comparison to multiplication and division.

To test any n, we wish to determine if $F_k(n)/n$ is the smallest that has yet been found. Let us assume $\hat{n}$ is value of n for which the smallest value of $F_k(n)/n$ has been obtained in all the n examined previous to the n which we are testing and $\hat{F} = F_k(\hat{n})$. We wish to determine if $F_k(n)n^{-1} < \hat{F}/\hat{n}$, or equivalently, if $F_k(n)\hat{n} < \hat{F}\cdot n$. Since we test the n in descending order, $F_k(n)\hat{n} < \hat{F}\cdot n$ only if $F_k(n) < \hat{F}$. From Lemma (2.2.2) and Diagram (3.1.2) we obtain:

(3.1.3)
$$F_k(n_1) = a\cdot 6^k + b(6^k+10^k)$$
$$F_k(n_2) = a\cdot 6^k + (2^k-b)\cdot 15^k$$
$$F_k(n_3) = (3^k-a)\cdot 10^k + c(6^k+10^k)$$
$$F_k(n_4) = (3^k-a)\cdot 10^k + (2^k-c)\cdot 15^k \quad .$$

Let $\hat{a} = \hat{F}_k/15^k$ and $\hat{b} = \hat{F}_k/10^k$. Then necessary conditions for $F_k(n) < \hat{F}$ are

(3.1.4)
$$b < \hat{b} \qquad (n = n_1)$$
$$(2^k-b) < \hat{a} \qquad (n = n_2)$$
$$(3^k-a) < \hat{b} \qquad (n = n_3)$$
$$(3^k-a) < \hat{b} \quad \text{and} \quad (2^k-c) < \hat{a} \qquad (n = n_4) \quad .$$

To test any n, we first perform the corresponding comparison from (3.1.4). If the comparison fails (which usually happens), we consider the next n. If the comparison succeeds, $F_k(n)$ is calculated from (3.1.3) and $F_k(n)\hat{n}$ is compared with $\hat{F}\cdot n$. If $F_k(n)\hat{n}$ is the smaller, $\hat{F}$, $\hat{n}$, $\hat{a}$ and $\hat{b}$ are replaced by $F_k(n)$, n, $F_k(n)/15^k$ and $F_k(n)/10^k$ respectively.

Remarks: 1) Using comparisons from (3.1.4), the value of n rarely has to be calculated, since these comparisons will usually fail. Thus n is not calculated except when the comparison for n fails. The value of n is calculated from the value of n_4, and (possibly) from some of the values of a, b and c. The values of n_4 is calculated from the value of $[6^k/5^k]\cdot 5^k$, which is stored in memory, and the number of times n_1 has been decremented (which is also kept track of).

2) The program initially sets $\hat{F}$ and $\hat{n}$, $\hat{a}$ and $\hat{b}$ so that the first n tested will result in $\hat{F}$ etc. being replaced.

When all the n have been tested (i.e., when $n_4 < \frac{1}{2} \cdot 6^k$), we must determine if n from Diagram (2.2.4) with $m > 0$ must be tested. Referring to (2.2.5), we see that $F(n)$ for any such n must satisfy $F(n) > m\cdot 18^k$. Thus if $\tilde{m} = [\frac{\hat{F}\, 6^k}{18^k\, \hat{n}}] \geq 1$ then we must test all n from Diagram (2.2.4), Case (i) or (ii) with $1 \leq m \leq \tilde{m}$. Otherwise we are done, and may output $\hat{n}$ as n_k. The testing of n when $m \geq 1$ follows a similar procedure to that which is given above. For any fixed value of m we test the n in descending sequence, with the difference that $\hat{F}(n)$ must now be

calculated on the basis that $m > 1$. We shall omit the precise details of this.

Remarks: 1) This algorithm is arranged to minimize the number of times the operation of multiplication is used, and in testing various specific n, division is never used. This minimizes processing time since multiplication and especially division are much slower than addition, subtraction and comparison.

2) As mentioned in the previous chapter, the calculated value of $\hat{E}$ was always sufficiently small that we could assume $m \leq 2$, and for $k \geq 38$, we could assume $m = 0$. This produced an enormous saving in time.

(§2) Numerical Results

The results are listed in three tables and two graphs. Table 1 gives the value of n_k and $q_k(n)$ for $2 \le n \le 75$. Table 2 gives the value of $1 - \exp\{-\frac{7}{48}(108)^k(E_k)^2\}$ for $5 \le n \le 75$. Table 3 gives the value of $(n_k/6^k)$ for $6 \le k \le 75$. Graph 1 is a graph of $g(t)$, the number of k $(5 \le k \le 75)$ for which $1 - \exp\{-\frac{7}{48}(108)^k(E_k)^2\} < x$ for $x = 0, .1, .2, \ldots, .9, 1.0$. This is plotted as triangles. The best fitting line for the data points (with $(0,0)$ being ignored) is also drawn. Graph 2 is a graph of $h(\mathbf{x},75)$ the number of k $(6 \le k \le 75)$ for which $(n_k/6^k) \le x$. This is plotted as triangles. The curve $70(\frac{8}{7}x^3 - \frac{1}{7}) = 80x^3 - 10$, which is the predicted value for the data, is also drawn.

CHAPTER IV

AN ASYMPTOTIC FORMULA FOR THE DENSITY OF THE (k,r)-FREE INTEGERS

(§1) Introduction

The principal results of this chapter shall be to obtain an asymptotic formula for $q_{(k;r)}(x)$, the number of (k,r)-free integers less than or equal to x. Two results are obtained in (§3), one without any hypotheses, and the other on the basis of the Riemann hypothesis.

Our results are similar to some results quoted in Chapter I. For analogous results for the (k,r) integers, we refer to Subbarao and Suryanarayana [19].

Before proving our main results, we require some lemmas.

(§2) Lemmas

We begin with a definition:

Definition (4.2.1): We define $\chi_{(k;r)}(n)$ by

(4.2.2) $$\chi_{(k;r)}(n) = \begin{cases} 0, & n \notin Q_{(k;r)} \\ 1, & n \in Q_{(k;r)} \end{cases}$$

where $Q_{(k;r)}$ is the set of all (k,r)-free integers. Thus $\chi_{(k;r)}(n)$ is the characteristic function of the (k,r)-free integers.

Remark: From Definition (4.2.1) it clearly follows that $q_{(k;r)}(x) = \sum\limits_{n \le x} \chi_{(k;r)}(n)$.

We now prove the fundamental lemma:

Lemma (4.2.3): (Fundamental Lemma)

(4.2.4) $$\chi_{(k;r)}(n) = \sum_{\substack{a^k b^r c = n \\ (a,b)=1 \\ a \in Q_2}} \mu(b)$$

(4.2.5) $$\chi_{(k;r)}(n) = \sum_{\substack{d^r \ell = n \\ (d^{k-r}, \ell) \in Q_{(k-r)}}} \mu(d)$$

where Q_ℓ is the set of ℓ-free integers ($\ell \geq 2$) and $\mu(n)$ is the well known Möbius μ function.

Proof: Since $\mu(n)$ is multiplicative, both

$$\sum_{\substack{a^k b^r c=n \\ (a,b)=1 \\ a\in Q_2}} \mu(b) \quad \text{and} \quad \sum_{\substack{d^r \ell=n \\ (d^{k-r},\ell)\in Q_{(k-r)}}} \mu(d) \quad \text{are multiplicative.}$$

Thus it is sufficient to prove Lemma (4.2.3) for $n = p^\alpha$, a prime power. But (4.2.4) and (4.2.5) are easily verified for $n = p^\alpha$, and the lemma follows. Q.E.D.

We require several results on the Möbius μ function.

Lemma (4.2.6): Define $M(x)$ as follows:

$$(4.2.7) \qquad M(x) = \sum_{n\leq x} \mu(n) \quad .$$

Then $M(x)$ satisfies

$$(4.2.8) \qquad M(x) = O(x\ \delta(x)) \quad \text{as} \quad x \to \infty$$

where

$$(4.2.9) \qquad \begin{aligned} \delta(x) &= \exp\{-A \log^{3/5} x(\log\log x)^{-(1/5)}\}, \quad x \geq 16 \\ &= \exp\{-2A\}\ , \quad 1 \leq x < 16 \end{aligned}$$

and A is an absolute positive constant. This result is due to Walfisz [21].

We now introduce $\phi_s(n)$, the generalized Jordan totient function, defined to be: $\phi_s(n) = n^s \prod_{p|n} (1-p^{-s})$ where s need not be an integer.

We use this function in the following lemma:

Lemma (4.2.10): Define $M(x,m)$ as follows:

$$(4.2.11) \qquad M(x,m) = \sum_{\substack{n \le x \\ (n,m)=1}} \mu(n)$$

then we have, for any fixed ϵ, $0 < \epsilon < 1$,

$$(4.2.12) \qquad M(x,m) = O(x \cdot \delta(x) \cdot m^{1-\epsilon}/\phi_{1-\epsilon}(m)) \quad \text{as} \quad x \to \infty$$

where the constant implied by the O-estimate depends only on ϵ

Proof: We claim that when x is sufficiently large, $\delta(x/s) \le s^{\epsilon}\delta(x)$.

It is equivalent to prove:

$$(x/s)^{\epsilon}\delta(x/s) \le x^{\epsilon}\,\delta(x) \quad .$$

To prove this, it is sufficient to prove $x^{\epsilon}\delta(x) \to \infty$ as $x \to \infty$ and that $\frac{d}{dx}(x^{\epsilon}\,\delta(x)) > 0$ for x sufficiently large. That $x^{\epsilon}\,\delta(x) \to \infty$ as $x \to \infty$ follows from the definition of $\delta(x)$.

$$\frac{d}{dx}(x^{\epsilon}\,\delta(x)) = x^{\epsilon-1}\,\delta(x)(\epsilon - A\log^{-(2/5)}x(\tfrac{3}{5}(\log\log x)^{-(1/5)} - \tfrac{1}{5}(\log\log x)^{-(6/5)}))$$

and since ϵ is fixed, $\epsilon > 0$, $\frac{d}{dx}(x^{\epsilon}\,\delta(x)) > 0$ for x sufficiently large. Thus for x sufficiently large, $\delta(\frac{x}{s}) \le s^{\epsilon}\,\delta(x)$.

Since replacing m by $\gamma(m)$, the largest square free divisor of m, does not alter either side of (4.2.12), we may, without loss of generality, assume m is square free. To prove the lemma, we shall induct on the number of prime divisors of m. If $m = 1$, (4.2.12) follows directly from Lemma (4.2.6). Assume Lemma (4.2.10) holds for m having $\le j$ prime divisors, and $m^* = p \cdot m$ has $(j+1)$

prime divisors. We have:

$$(4.2.13)\qquad M(x,m^*) = M(x,p\cdot m)$$

$$= \sum_{\substack{n\le x\\(n,m)=1}} \mu(n) - \sum_{\substack{n\le x\\p\mid n\\(n,m)=1}} \mu(n)$$

$$= M(x,m) - \sum_{\substack{p\cdot n\le x\\(n,m)=1}} \mu(p\cdot n)$$

$$= M(x,m) + \sum_{\substack{n\le x/p\\(n,p)=1\\(n,m)=1}} \mu(n)$$

$$= M(x,m) + M(x/p,\ m^*)\quad .$$

Iterating (4.2.13), we find

$$(4.2.14)\qquad M(x,m^*) = \sum_{k=o}^{c} M(x/p^k,\ m)$$

where $c = [\log x/\log p]$. By the inductive hypothesis and (4.2.12), we have:

$$M(x/p^k,\ m) = O((x/p^k)\ \delta(x/p^k)\ m^{1-\epsilon}/\phi_{1-\epsilon}(m))\quad .$$

But we may assume $\delta(x/p^k) < p^{k\epsilon}\cdot\delta(x)$. Thus

$$(4.2.15)\qquad M(x/p^k,\ m) = O((x/p^{k-k\epsilon})\,\delta(x)\cdot m^{1-\epsilon}/\phi_{1-\epsilon}(m))\quad .$$

Combining (4.2.14) and (4.2.15) we find:

$$M(x,m^*) = O\left(\sum_{k=o}^{c} (x/p^{k-k\epsilon})\,\delta(x)\ m^{1-\epsilon}/\phi_{1-\epsilon}(m)\right)$$

$$= O\left(x\cdot\delta(x)\cdot(m^{1-\epsilon}/\phi_{1-\epsilon}(m))\cdot \sum_{k=o}^{\infty} p^{-k(1-\epsilon)}\right)$$

$$= O\left(x\cdot\delta(x)\cdot(m^{1-\epsilon}/\phi_{1-\epsilon}(m))\cdot (p^{1-\epsilon}/p^{1-\epsilon}-1)\right)$$

$$= O\left(x\cdot\delta(x)\cdot(m^*)^{1-\epsilon}/\phi_{1-\epsilon}(m^*)\right) .$$

Thus (4.2.12) holds for m^* and Lemma (4.2.10) follows by induction. Q.E.D.

Lemma (4.2.16): For ϵ fixed, $0 < \epsilon < 1$, and $k \geq 2$ and $z > 0$, we have:

$$(4.2.17) \qquad \sum_{\substack{b>z \\ (b,a)=1}} \mu(b)/b^k = O(z^{-(k-1)}\delta(z)\ a^{1-\epsilon}/\phi_{1-\epsilon}(a)) \quad \text{as} \quad z \to \infty$$

where the constant implied by the O-estimate depends only on ϵ.

Proof:

$$\sum_{\substack{b>z \\ (b,a)=1}} \mu(b)/b^k = k \sum_{\substack{b>z \\ (b,a)=1}} \mu(b) \int_b^\infty y^{-(k+1)}\,dy$$

$$= k \int_z^\infty y^{-(k+1)} \left(\sum_{\substack{z<b\leq y \\ (b,a)=1}} \mu(b)\right) dy$$

$$= k \int_z^\infty y^{-(k+1)} \left(\sum_{\substack{b\leq y \\ (b,a)=1}} \mu(b)\right) dy \ -$$

$$-k \int_z^\infty y^{-(k+1)} \Big(\sum_{\substack{b \le z \\ (b,a)=1}} \mu(b) \Big) dy$$

(4.2.18) $$= k \int_z^\infty M(y,a) y^{-(k+1)} dy - kM(z,a) \int_z^\infty y^{-(k+1)} dy \quad .$$

Applying (4.2.12) to (4.2.18), we have:

$$\sum_{\substack{b>a \\ (b,a)=1}} \mu(b)/b^k = O(k \int_z^\infty \delta(y) \cdot (a^{1-\epsilon}/\phi_{1-\epsilon}(a)) y^{-k} dy)$$
$$+ O(\delta(z)(a^{1-\epsilon}/\phi_{1-\epsilon}(a)) z^{-(k-1)}) \quad .$$

Since δ is monotonically decreasing, we may replace $\delta(y)$ in the first O-estimate above to obtain:

$$\sum_{\substack{b>z \\ (b,a)=1}} \mu(b)/b^k = O(\delta(z)(a^{1-\epsilon}/\phi_{1-\epsilon}(a)) \int_z^\infty ky^{-k} dy)$$
$$+ O(\delta(z)(a^{1-\epsilon}/\phi_{1-\epsilon}(a)) z^{-(k-1)})$$
$$= O(\delta(z) \cdot (a^{1-\epsilon}/\phi_{1-\epsilon}(a)) z^{-(k-1)})$$

Q.E.D.

Lemma (4.2.19): (Cf., Titchmarsh [20], Theorem 14-26(A), p. 316). If the Riemann hypothesis is true, then for $x \geq 1$,

(4.2.20) $$M(x) = O(x^{1/2}\, \omega(x)) \quad \text{as} \quad x \to \infty$$

where

$$(4.2.21) \qquad \omega(x) = \exp\{A \log x(\log\log x)^{-1}\}\ , \qquad x \geq 16$$

$$= \exp\{A \log 16\ (\log\log 16)^{-1}\}\ , \qquad x < 16\ .$$

and A is an absolute positive constant.

Lemma (4.2.22): If the Riemann hypothesis is true, then for $x \geq 1$,

$$(4.2.23) \qquad M(x,m) = O(x^{1/2}\ \omega(x)\ m^{1/2}/\phi_{1/2}(m)) \qquad \text{as} \quad x \to \infty$$

where the constant implied by the O-estimate is independent of m and x.

Proof: We employ an inductive argument similar to the one for Lemma (4.2.10). Again, we may assume m is square free. If $m = 1$, Lemma (4.2.22) follows directly from Lemma (4.2.19). Let us assume Lemma (4.2.22) holds for m having $\leq j$ prime divisors and let $m^* = m \cdot p$ have $(j+1)$ prime divisors. Then, as in Lemma (4.2.10),

$$M(x,m^*) = \sum_{k=o}^{c} M(x/p^k,\ m)$$

where $c = [\log x/\log p]$. By the inductive hypothesis, we find:

$$M(x,m^*) = O\left\{\sum_{k=o}^{c} ((x/p)^{1/2}\ \omega(x/p)\cdot m^{1/2}/\phi_{1/2}(m))\right\}$$

$$= O(x^{1/2}\ \omega(x)\,(m^{1/2}/\phi_{1/2}(m))\ \sum_{k=o}^{\infty} p^{-(1/2)})$$

since $\omega(x)$ is monotonically increasing. Thus

$$M(x,m^*) = O(x^{1/2}\ \omega(x)\,(m^{1/2}/\phi_{1/2}(m))\cdot p^{1/2}/p^{1/2} - 1)$$

$$= O(x^{1/2}\ \omega(x)\,(m^*)^{1/2}/\phi_{1/2}(m^*))$$

and the lemma follows by induction. Q.E.D.

Lemma (4.2.24): Given $z \geq 1$, $k \geq 2$, and the Riemann hypothesis then:

$$(4.2.25) \qquad \sum_{\substack{b>z \\ (b,a)=1}} \mu(b)/b^k = O(\omega(z)\cdot z^{-(1/2)}\cdot z^{-(k-1)}\cdot a^{1/2}/\phi_{1/2}(a))$$

as $z \to \infty$

where the constant implied by the O-estimate is independent of z and a.

Proof: Since a straight foreward differentiation shows $\omega(z)\cdot z^{-(1/2)}$ is monotonically decreasing for z sufficiently large, we follow the proof of Lemma (4.2.16), replacing $\delta(z)$ by $\omega(z)\cdot z^{-(1/2)}$, using Lemma (4.2.22) in place of Lemma (4.2.10) and easily derive (4.2.25).

Q.E.D.

(§3) The Main Results

Theorem (4.3.1): For $2 \leq r < k \leq \infty$, we have:

$$q_{(k;r)}(x) = x \cdot c_{(k;r)} + O(x \cdot \delta_r(x)^{\frac{k}{k-r}}) \quad \text{as} \quad x \to \infty \tag{4.3.2}$$

where

$$c_{(k;r)} = \prod_p (1 - p^{-r} + p^{-k}) \tag{4.3.3}$$

and

$$\delta_r(x) = \exp(-c_r (\log^{3/5} x)(\log \log x)^{-(1/5)}) \tag{4.3.4}$$

where c_r depends only on r and the O-estimate is uniform in k and r.

Proof: From Lemma (4.2.3) it follows that:

$$q_{(k;r)}(x) = \sum_{n \leq x} \chi_{(k;r)}(n)$$

$$= \sum_{\substack{a^k b^r c \leq x \\ (a,b)=1 \\ a \in Q_2}} \mu(b) \quad . \tag{4.3.5}$$

Let $z = x^{1/r}$ and $\rho = \rho(z)$ be a function of z, $\rho < 1$, such that $(\rho \cdot z) \to \infty$ as $z \to \infty$ (ρ will be chosen later). Then if $a^k b^r c \leq x$, we cannot have both $b > \rho z$ and $a^k c > \rho^{-r}$. Thus we have:

$$(4.3.6)\qquad q_{(k;r)}(x) = \sum_{\substack{a^k b^r c \le x \\ (a,b)=1 \\ a\in Q_2 \\ b<\rho z}} \mu(b) + \sum_{\substack{a^k b^r c\le x \\ (a,b)=1 \\ a\in Q_2 \\ a^k c \le \rho^{-r}}} \mu(b) - \sum_{\substack{b<\rho z \\ a^k c\le \rho^{-r} \\ (a,b)=1 \\ a\in Q_2}} \mu(b)$$

$$= S_1 + S_2 + S_3 \qquad \text{(say)} \quad .$$

We consider each of these sums separately

$$S_1 = \sum_{b<\rho z} \mu(b) \sum_{\substack{a^k c\le x/b^r \\ (a,b)=1 \\ a\in Q_2}} 1$$

$$= \sum_{b<\rho z} \mu(b) \sum_{\substack{a^k \le x/b^r \\ (a,b)=1 \\ a\in Q_2}} [x/a^k b^r]$$

$$= \sum_{b<\rho z} \mu(b) \sum_{\substack{a^k \le x/b^r \\ (a,b)=1 \\ a\in Q_2}} ((x/a^k b^r)+0(1))$$

$$= \sum_{b<\rho z} \mu(b) \Big\{0(x^{1/k}/b^{r/k}) + \sum_{\substack{(a,b)=1 \\ a\in Q_2}} (x/a^k b^r)$$

$$+ 0\Big(\sum_{a=[x^{1/k}/b^{r/k}]}^{\infty} (x/a^k b^r)\Big)\Big\}$$

$$= x \sum_{b<\rho z} \mu(b)/b^r \sum_{\substack{(a,b)=1 \\ a\in Q_2}} 1/a^k + 0(x^{1/k} \sum_{b=1}^{\rho z} b^{-r/k})$$

$$+ 0\Big(\sum_{b<\rho z} \sum_{a=[x^{1/k}/b^{r/k}]}^{\infty} (x/a^k b^r)\Big)$$

A simple calculation shows, since $x^{1/r} = z$, we have:

$$O(x^{1/k} \sum_{b=1}^{\rho z} 1/b^{r/k}) = O(z \cdot \rho^{1-r/k} \cdot \frac{k}{k-r})$$

and

$$O\Big(\sum_{b<\rho z} \sum_{a=[x^{1/k}/b^{r/k}]}^{\infty} (x/a^k b^r)\Big) = O(z \cdot \rho^{1-r/k} \cdot \frac{k}{k-r})$$

Thus

$$(4.3.7) \quad S_1 = x \cdot \sum_{b=1}^{\infty} (\mu(b)/b^r) \sum_{\substack{(a,b)=1 \\ a\in Q_2}} 1/a^k$$

$$+ O(x \sum_{b\geq \rho z} (\mu(b)/b^r) \sum_{\substack{(a,b)=1 \\ a\in Q_2}} 1/a^k) + O(z \rho^{1-\frac{r}{k}} \cdot \frac{k}{k-r})$$

We now examine the first O-term more closely:

$$(4.3.8) \quad x \sum_{b\geq \rho z} (\mu(b)/b^r) \sum_{\substack{(a,b)=1 \\ a\in Q_2}} 1/a^k = x \sum_{a\in Q_2} a^{-k} \sum_{\substack{b\geq \rho z \\ (b,a)=1}} \mu(b)/b^r .$$

Applying Lemma (4.2.16) to (4.3.8) with $\epsilon = 1/2$, we have

$$x \sum_{b\geq \rho z} (\mu(b)/b^r) \sum_{\substack{(a,b)=1 \\ a\in Q_2}} a^{-k} = x \sum_{a\in Q_2} O(\delta(\rho z)$$

$$\cdot\, a^{(1/2)-k}/\rho^{r-1} z^{r-1} \phi_{1/2}(a))$$

$$= O(x \cdot \delta(\rho z) \cdot \rho^{1-r} \cdot x^{-1} \cdot z \sum_{a=1}^{\infty} (a^{(1/2)-k}/\phi_{1/2}(a))$$

$$(4.3.9) \quad = O(z\ \delta(\rho z) \rho^{1-r})$$

since $\sum_{a=1}^{\infty} (a^{(1/2)-k}/\phi_{1/2}(a))$ converges uniformly for $k \geq 3$. Thus by (4.3.7) and (4.3.9), we have:

$$S_1 = x \sum_{b=1}^{\infty} (\mu(b)/b^r) \sum_{\substack{(a,b)=1 \\ a\in Q_2}} 1/a^k + 0(z\delta(\rho z)\rho^{1-r}) + 0(z\rho^{1-r/k} \cdot \frac{k}{k-r})$$

$$(4.3.10) \quad = x \cdot c_{(k;r)} + 0(z\ \delta(\rho z)\rho^{1-r}) + 0(z\ \rho^{1-r/k} \cdot \frac{k}{k-r}) ,$$

by a simple argument and (4.3.3). We next consider S_2:

$$S_2 = \sum_{\substack{(a,b)=1 \\ a\in Q_2 \\ a^k c\leq\rho^{-r} \\ a^k b^r c\leq x}} \mu(b)$$

$$(4.3.11) \quad = \sum_{\substack{a^k c\leq\rho^{-r} \\ a\in Q_2}} \ \sum_{\substack{b\leq(x/a^k c)^{1/r} \\ (a,b)=1}} \mu(b) .$$

Applying Lemma (4.2.10) to the inner sum of (4.3.11) we obtain (again with $\epsilon = 1/2$):

$$S_2 = \sum_{\substack{a^k c\leq\rho^{-r} \\ a\in Q_2}} 0\{(x/a^k c)^{1/r}\ \delta((x/a^k c)^{1/r})\ a^{1/2}/\phi_{1/2}(a)\}$$

$$(4.3.12) \quad = 0\{x^{1/r} \sum_{\substack{a^k c\leq\rho^{-r} \\ a\in Q_2}} (a^k c)^{-(1/r)}\ \delta((x/a^k c)^{1/r})\, a^{1/2}/\phi_{1/2}(a)\} .$$

But we have, since $\delta(x)$ is monotonically decreasing, and $a^k c \leq \rho^{-r}$, that $(1/a^k c)^{1/r} \geq \rho$ and $\delta((x/a^k c)^{1/r}) \leq \delta(\rho x^{1/r}) = \delta(\rho z)$. Thus, from (4.3.12) we have:

$$S_2 = O\{ x^{1/r} \delta(\rho z) \sum_{a \leq \rho^{-r/k}} a^{-k/r}(a^{1/2}/\phi_{1/2}(a)) \sum_{c \leq \rho^{-r} a^{-k}} c^{-1/r}\}$$

$$(4.3.13) \quad = O\{z\, \delta(\rho z) \sum_{a \leq \rho^{-r/k}} a^{-k/r}(a^{1/2}/\phi_{1/2}(a))(\rho^{-r} a^{-k})^{1-(1/r)}\}$$

$$S_2 = O\{z\, \delta(\rho z)\rho^{1-r} \sum_{a \leq \rho^{-r/k}} a^{-k}\, a^{1/2}/\phi_{1/2}(a)\}$$

$$(4.3.14) \quad = O(z\, \delta(\rho z)\rho^{1-r})$$

since $\sum_{a=1}^{\infty} a^{-k}\, a^{1/2}/\phi_{1/2}(a)$ converges uniformly for $k \geq 3$. Finally, we have:

$$S_3 = \sum_{\substack{b < \rho z \\ a^k c < \rho z \\ (a,b)=1 \\ a \in Q_2}} \mu(b)$$

$$(4.3.15) \quad = \sum_{\substack{a^k c \leq \rho^{-r} \\ a \in Q_2}} \sum_{\substack{b < \rho z \\ (a,b)=1}} \mu(b) \quad .$$

Applying Lemma (4.2.10) to (4.3.15), we obtain for $\epsilon = 1/2$:

$$S_3 = \sum_{\substack{a^k c \leq \rho^{-r} \\ a \in Q_2}} O\{\rho z\, \delta(\rho z)\, a^{1/2}/\phi_{1/2}(a)\}$$

$$= 0\{ \sum_{\substack{a \le \rho^{-r/k} \\ a \in Q_2}} \rho z \cdot \delta(\rho z) \cdot (a^{1/2}/\phi_{1/2}(a)) \cdot \rho^{-r}/a^k \}$$

$$= 0\ \{z \cdot \delta(\rho z) \cdot \rho^{1-r} \sum_{\substack{a \le \rho^{-r/k} \\ a \in Q_2}} a^{-k}\ a^{1/2}/\phi_{1/2}(a)\}$$

$$(4.3.16) \qquad = \quad 0(z \cdot \delta(\rho z)\rho^{1-r})$$

since $\sum_{a=1}^{\infty} a^{-k}\ a^{1/2}/\phi_{1/2}(a)$ converges uniformly for $k \ge 3$.

Thus, substituting (4.3.10), (4.3.14) and (4.3.16) into (4.3.6), we have:

$$(4.3.17) \qquad q_{(k;r)}(x) = x \cdot c_{(k;r)} + 0(z \cdot \delta(\rho z) \cdot \rho^{1-r}) + 0(z \cdot \rho^{1-(r/k) \cdot \frac{k}{k-r}})$$

Setting $\rho = \rho(x) = (\delta(x^{1/2r}))^{1/r}$, from (4.3.17) we obtain (4.3.2). The calculations follow the procedure of Subbarao and Suryanarayana [19]. Q.E.D.

<u>Theorem (4.3.18)</u>: If $2 \le r < k \le \infty$ and the Riemann hypothesis is true, then:

$$(4.3.19) \qquad q_{(k;r)}(x) = x \cdot c_{(k;r)} + 0(x^{1/r}\ \omega(x) x^{-((k-r)/r(2kr+k-2r) \cdot \frac{k}{k-r}})$$

as $x \to \infty$

where $\omega(x)$ is given by (4.2.21), and the 0-estimate is uniform in k and r.

Proof: Following the proof of Theorem (4.3.1), replacing $\delta(x)$ by $(\omega(x)\ x^{-(1/2)})$, (which is, as we remarked before, monotonically decreasing for x sufficiently large), and using Lemmas (4.2.22) and (4.2.24) in place of Lemmas (4.2.12) and (4.2.16) respectively, we obtain:

$$q_{(k;r)}(x) = x\cdot c_{(k;r)} + O(z\cdot\omega(\rho z)\cdot(\rho z)^{-(1/2)}\cdot\rho^{1-r})$$

$$+ O(z\cdot\rho^{1-(r/k)}\cdot\frac{k}{k-r})$$

$$= x\cdot c_{(k;r)} + O(z^{1/2}\ \omega(x)\rho^{(1/2)-r}) + O(z\cdot\rho^{1-(r/k)}\cdot\frac{k}{k-r}) \tag{4.3.20}$$

Setting $\rho = z^{-1/(1+2r-(2r/k))}$, we obtain (4.3.19) from (4.3.20). Q.E.D.

Remark. The proofs of Theorems (4.3.1) and (4.3.18) when $k = \infty$ requires a minor modification of the proofs given above.

CHAPTER V

THE SCHNIRELMANN DENSITY OF THE (k,r)-FREE INTEGERS AND THE (k,r) INTEGERS

(§1) Introduction

In this chapter we extend the result of Orr [12] to the (k,r)-free integers and to the (k,r) integers. In fact, it will be proved by completely elementary methods that for all $2 \leq r < k \leq \infty$, the Schnirelmann density of the (k,r) integers, which we denote by $d_{k,r}$, is less than their asymptotic density, which we denote by $D_{k,r}$. We prove that the same result, with the possible exception of $r = 2$ & $k = 3$, holds for the (k,r)-free integers. We also show that for $3 \leq r < k \leq \infty$ in the above results, the Schnirelmann densities are achieved somewhere on the interval $[2^r, 8^r)$. We improve this interval for most values of k. We also show that for most values of k and r (except $(k,r) = (2,3)$) $d_{k,r} = d_{(k;r)}$ and are achieved at the same point.

It should be noted that a proof that $d_{k,r} < D_{k,r}$ was given by Subbarao and Feng [17]. However, their proof was not completely elementary, and it gave no indication of where the Schnirelmann density could be realized. Before proving our main result, we require some lemmas. In (§2), we calculate an exact bound for the quantities $|q_{(k;r)}(n) - n \cdot D_{(k;r)}|$ and $|q_{k,r}(n) - n \cdot D_{k,r}|$. In (§3), we calculate bounds for $d_{(k;r)}$ under various conditions for k and r. In (§4), certain algebraic results required in our main

theorem are proved. In (§5), our main results (stated above) are obtained.

Let us first summarize the notation already introduced and introduce some more:

$d_{(k;r)}$ = the Schnirelmann density of the (k,r)-free integers.

$d_{k,r}$ = the Schnirelmann density of the (k,r) integers.

$D_{(k;r)}$ = the asymptotic density of the (k,r)-free integers.

$D_{k,r}$ = the asymptotic density of the (k,r) integers.

$Q_{(k;r)}$ = the set of all (k,r)-free integers.

$Q_{k,r}$ = the set of all (k,r) integers.

$$\chi_{(k;r)}(n) = \begin{cases} 1, & n \in Q_{(k;r)} \\ 0, & n \notin Q_{(k;r)} \end{cases} .$$

$$\chi_{k,r}(n) = \begin{cases} 1, & n \in Q_{k,r} \\ 0, & n \notin Q_{k,r} \end{cases} .$$

$$q_{(k;r)}(x) = \sum_{n \leq x} \chi_{(k;r)}(n) .$$

$$q_{k,r}(x) = \sum_{n \leq x} \chi_{k,r}(n) .$$

(§2) Numerical Bounds for $E_{k,r}$ and $E_{(k;r)}$

Let us recall, from Chapter IV, that $D_{(k;r)} = c_{(k;r)} = \prod_p (1 - p^{-r} + p^{-k})$. Also, as was shown by Feng and Subbarao [18]. $D_{k,r} = \zeta(k)/\zeta(r)$, where ζ is the Riemann zeta function.

The two lemmas which we prove in this section are extensions of Lemma 2 in Orr's paper [12]. They provide an explicit numerical bound for the difference between $q_{(k;r)}(n)$ and $n \cdot D_{(k;r)}$ (in the case of the (k,r)-free integers) and between $q_{k,r}(n)$ and $n \cdot D_{k,r}$ (in the case of the (k,r) integers).

Lemma (5.2.1). Let $2 \leq r < k \leq \infty$, and define $E_{(k;r)}(n)$ by

(5.2.2) $$E_{(k;r)}(n) = q_{(k;r)}(n) - n \cdot c_{(k;r)} .$$

Then $E_{(k;r)}(n)$ satisfies the following inequalities:

(5.2.3) $$|E_{(k;r)}(n)| \leq \left\{\frac{k}{(k-r)\zeta(2)} + \frac{2k}{2k-r} + \frac{k}{(k-r)(k-1)} + 1 + \right.$$
$$\left. + \frac{\zeta(k)}{r-1} \right\} n^{1/r} + \zeta(k)$$

(5.2.4) $$|E_{(k;r)}(n)| \leq \left\{(\gamma + \frac{1}{r} \log n)(\frac{1}{\zeta(2)} + \frac{1}{r-1}) + 4 + \right.$$
$$\left. + \frac{\zeta(k)}{r-1} \right\} n^{1/r} + \zeta(k) \quad .$$

Lemma (5.2.5). Let $2 \leq r < k \leq \infty$, and define $E_{k,r}(n)$ by

(5.2.6) $$E_{k,r}(n) = q_{k,r}(n) - n \cdot \{\frac{\zeta(k)}{\zeta(r)}\} \quad .$$

Then $E_{k,r}(n)$ satisfies the following inequalities:

(5.2.7) $$|E_{k,r}(n)| \leq \frac{k}{k-r} \left(\frac{6}{\pi^2} + \frac{1}{r-1}\right) n^{1/r} + \frac{n^{1/k}}{k-1} + 2n^{\frac{1}{2}(\frac{1}{r} + \frac{1}{k})}$$

(5.2.8) $$|E_{k,r}(n)| \leq \{\frac{1}{r} ((\frac{6}{\pi^2} + \frac{1}{r-1}) \log n) + 4\} n^{1/r} \quad .$$

<u>Proof of Lemma (5.2.1):</u> By Lemma (4.2.3), we have

$$q_{(k;r)}(n) = \sum_{\substack{a^k b^r c \leq n \\ (a,b)=1 \\ a \in Q_2}} \mu(b)$$

$$= \sum_{b} \mu(b) \sum_{\substack{a^k c \leq n/b^r \\ (a,b)=1 \\ a \in Q_2}} 1$$

$$= \sum_{b} \mu(b) \sum_{\substack{a \in Q_2 \\ (a,b)=1}} \left[\frac{n}{a^k b^r}\right]$$

$$= n \cdot \sum_{b} \mu(b) \sum_{\substack{a \in Q_2 \\ (a,b)=1}} \frac{1}{a^k b^r} - \sum_{b} \mu(b) \cdot$$

$$\sum_{\substack{a \in Q_2 \\ (a,b)=1}} \left(\frac{n}{a^k b^r} \right) - \left[\frac{n}{a^k b^r}\right]$$

$$= n\cdot c_{(k;r)} - \sum_{b} \mu(b) \sum_{\substack{a\in Q_2 \\ (a,b)=1}} \left(\frac{n}{a^k b^r}\right) - \left[\frac{n}{a^k b^r}\right] .$$

Thus we have

$$E_{(k;r)}(n) = \sum_{b} \mu(b) \sum_{\substack{a\in Q_2 \\ (a,b)=1}} \left[\frac{n}{a^k b^r}\right] - \left(\frac{n}{a^k b^r}\right) ,$$

and we have

$$|E_{(k;r)}(n)| \leq \sum_{b\in Q_2} \sum_{\substack{a\in Q_2 \\ (a,b)=1}} \left| \left[\frac{n}{a^k b^r}\right] - \left(\frac{n}{a^k b^r}\right) \right|$$

$$\leq \sum_{b} \sum_{a\in Q_2} \left(\frac{n}{a^k b^r}\right) - \left[\frac{n}{a^k b^r}\right] . \tag{5.2.9}$$

We use the following results:

If $a^k b^r \leq n$, then $\left(\frac{n}{a^k b^r}\right) - \left[\frac{n}{a^k b^r}\right] < 1$, and if $a^k b^r \geq n$,

then $\left(\frac{n}{a^k b^r}\right) - \left[\frac{n}{a^k b^r}\right] \leq \left(\frac{n}{a^k b^r}\right)$. We have

$$|E_{(k;r)}(n)| \leq \sum_{b<n^{1/r}} \sum_{\substack{a\in Q_2 \\ a\leq\left(\frac{n}{b^r}\right)^{1/k}}} 1 + \sum_{b<n^{1/r}} \sum_{\substack{a\in Q_2 \\ a>\left(\frac{n}{b^r}\right)^{1/k}}} 1 \tag{5.2.10}$$

$$\cdot \frac{n}{a^k b^r} + \sum_{b\geq n^{1/r}} \sum_{a\in Q_2} \frac{n}{a^k b^r}$$

We shall estimate each of the three sums on the RHS of (5.2.10). By a result of Moser and MacLeod [11]:

$$\sum_{b<n^{1/r}} \sum_{\substack{a\le(\frac{n}{b^r})^{1/k} \\ a\in Q_2}} 1 \le \sum_{b<n^{1/r}} \left\{ \frac{(\frac{n}{b^r})^{1/k}}{\zeta(2)} + (\frac{n}{b^r})^{1/2k} \right\}$$

$$= \frac{n^{1/k}}{\zeta(2)} \sum_{b<n^{1/r}} b^{-r/k} + n^{1/2k} \cdot \sum_{b<n^{1/r}} b^{-r/2k}$$

$$< \frac{n^{1/k}}{\zeta(2)} \int_0^{n^{1/r}} x^{-r/k} dx + n^{1/2k} \int_0^{n^{1/r}} x^{-r/2k} dx$$

(5.2.11) $$= \frac{n^{1/r}}{\zeta(2)(1 - \frac{r}{k})} + \frac{n^{1/r}}{1 - \frac{r}{2k}} .$$

We examine the second sum from (5.2.10), to find

$$\sum_{b<n^{1/r}} \sum_{\substack{a\in Q_2 \\ a>(\frac{n}{b^r})^{1/k}}} \frac{n}{a^k b^r} \le n \sum_{b<n^{1/r}} \frac{1}{b^r} \sum_{a>(\frac{n}{b^r})^{1/k}} \frac{1}{a^k}$$

$$\le n \sum_{b<n^{1/r}} \frac{1}{b^r} \left(\frac{b^r}{n} + \int_{(\frac{n}{b^r})^{1/k}}^{\infty} x^{-k} dx \right)$$

$$\le \sum_{b<n^{1/r}} 1 + \frac{n^{1/k}}{k-1} \sum_{b<n^{1/r}} b^{-r/k}$$

$$\le n^{1/r} + \frac{n^{1/k}}{k-1} \int_0^{n^{1/r}} x^{-r/k} dx$$

$$= n^{1/r} + \frac{n^{1/r}}{(k-1)(1 - r/k)} \quad . \tag{5.2.12}$$

Finally, the third sum in (5.2.10) yields:

$$\sum_{b \geq n^{1/r}} \sum_{a \in Q_2} \frac{n}{a^k b^r} \leq n \sum_{b \geq n^{1/r}} b^{-r} \sum_a a^{-k}$$

$$\leq n \cdot \zeta(k) \left(\frac{1}{n} + \int_{n^{1/r}}^{\infty} x^{-r} dx\right)$$

$$= \zeta(k) + \frac{\zeta(k) \cdot n^{1/r}}{r-1} \quad . \tag{5.2.13}$$

Substituting (5.2.11), (5.2.12) and (5.2.13) into (5.2.10), we obtain (5.2.3), proving the first half of Lemma (5.2.1).

Let us now estimate the first two sums in (5.2.10) in a different manner. From the first sum, we obtain:

$$\sum_{b < n^{1/r}} \sum_{\substack{a \leq (\frac{n}{b^r})^{1/k} \\ a \in Q_2}} 1 < \sum_{b < n^{1/r}} \sum_{\substack{a \leq (\frac{n}{b^r})^{1/r} \\ a \in Q_2}} 1 \quad .$$

Applying the same result of Moser and MacLeod [11] we have:

$$\sum_{b < n^{1/r}} \sum_{\substack{a \leq (\frac{n}{b^r})^{1/k} \\ a \in Q_2}} 1 < \sum_{b < n^{1/r}} \left[\frac{(\frac{n}{b^r})^{1/r}}{\zeta(2)} + (\frac{n}{b^r})^{1/2r} \right]$$

$$= \frac{n^{1/r}}{\zeta(2)} \sum_{b < n^{1/r}} \frac{1}{b} + n^{1/2r} \sum_{b < n^{1/r}} b^{-1/2}$$

$$< \frac{n^{1/r}}{\zeta(2)} (\gamma + \frac{1}{r} \log n) + n^{1/2r} \int_o^{n^{1/r}} x^{-1/2} dx$$

$$\frac{n^{1/r}}{\zeta(2)} (\gamma + \frac{1}{r} \log n) + 2n^{1/r} . \tag{5.2.14}$$

We next estimate the second sum from (5.2.10). It can easily be seen that for $k > r$, $\sum_{a>x^{1/k}} 1/a^k < \frac{1}{x} + \sum_{a>x^{1/r}} 1/a^r$.

Thus, for the second sum, we find

$$n \sum_{b<n^{1/r}} \frac{1}{b^r} \sum_{a>(\frac{n}{b^r})^{1/k}} 1/a^k < n \sum_{b<n^{1/r}} \frac{1}{b^r} \left(\frac{b^r}{n} + \sum_{a>(\frac{n}{b^r})^{1/r}} 1/a^r \right)$$

$$\leq \sum_{b<n^{1/r}} 1 + n \sum_{b<n^{1/r}} \frac{1}{b^r} \left(\frac{b^r}{n} + \int_{(\frac{n}{b^r})^{1/r}}^{\infty} \frac{1}{x^r} dx \right)$$

$$\leq 2 \sum_{b<n^{1/r}} 1 + \frac{n^{1/r}}{r-1} \sum_{b<n^{1/r}} 1/b$$

$$< 2 n^{1/r} + \frac{n^{1/r}}{r-1} \left(\frac{\log n}{r} + \gamma \right) \tag{5.2.15}$$

Substituting (5.2.14), (5.2.15) and (5.2.13) into (5.2.10), we obtain (5.2.4). Q.E.D.

<u>Proof of Lemma (5.2.5)</u>: Let $N = [n^{1/k}]$. Then Feng [9] has shown that:

(5.2.16) $$q_{k,r}(n) = q_r(\frac{n}{1^k}) + q_r(\frac{n}{2^k}) + \ldots + q_r(\frac{n}{N^k}) \ .$$

Let

(5.2.17) $$E_r(x) = q_r(x) - \frac{x}{\zeta(r)} \ .$$

Then

$$q_{k;r}(n) = \frac{n}{\zeta(r)} \left(\frac{1}{1^k} + \frac{1}{2^k} + \ldots + \frac{1}{N^k} \right) + E_r\left(\frac{n}{1^k}\right) +$$

$$+ E_r(\frac{n}{2^k}) + \ldots + E_r(\frac{n}{N^k}) \ .$$

(5.2.18) $$= n \cdot \frac{\zeta(k)}{\zeta(r)} - \frac{n}{\zeta(r)} \sum_{j=N+1}^{\infty} j^{-k} + \{E_r(\frac{n}{1^k}) + \ldots + E_r(\frac{n}{N^k})\} \ .$$

Orr [12] proved that

(5.2.19) $$|E_r(x)| \leq \left(\frac{6}{\pi^2} + \frac{1}{r-1} + \frac{1}{x^{1/2r}}\right) x^{1/r} \ .$$

Applying (5.2.19) to (5.2.18), we have:

(5.2.20) $$|E_{k,r}(n)| \leq \left\{\frac{n}{\zeta(r)} \sum_{j=N+1}^{\infty} \frac{1}{j^k}\right\} + \sum_{j=1}^{N} \left(\frac{6}{\pi^2} + \frac{1}{r-1} + \right.$$

$$\left. + \frac{1}{(n/j^k)^{1/2r}}\right) \left(\frac{n}{j^k}\right)^{1/r} \ .$$

Now we estimate $\sum_{j=N+1}^{\infty} j^{-k}$:

$$\sum_{j=N+1}^{\infty} j^{-k} \le \frac{1}{(N+1)^k} + \int_{N+1}^{\infty} u^{-k}\, du$$

$$(5.2.21) \qquad = \frac{1}{(N+1)^k} + \frac{1}{k-1} \cdot \frac{1}{(N+1)^{k-1}} \quad .$$

But $N = [n^{1/k}]$, so $(N+1) > n^{1/k}$ and $(N+1)^k > n$. Therefore

$$\frac{n}{\zeta(r)} \sum_{j=N+1}^{\infty} j^{-k} \le \frac{1}{\zeta(r)} \left(1 + \frac{n^{1/k}}{k-1}\right)$$

$$(5.2.22) \qquad \le 1 + \frac{n^{1/k}}{k-1} \quad .$$

We now estimate the second sum in (5.2.20).

$$\sum_{j=1}^{N} \left(\frac{6}{\pi^2} + \frac{1}{r-1} + \frac{1}{(n/j^k)^{1/2r}}\right) \left(\frac{n}{j^k}\right)^{1/r} =$$

$$(5.2.23) \qquad = \left(\frac{6}{\pi^2} + \frac{1}{r-1} + \frac{1}{n^{1/2r}}\right) n^{1/r} + \sum_{j=2}^{N} \left(\frac{n}{j^k}\right)^{1/2r} +$$

$$+ \left(\frac{6}{\pi^2} + \frac{1}{r-1}\right) \sum_{j=2}^{N} \left(\frac{n}{j^k}\right)^{1/r} \quad .$$

Now we have

$$\sum_{j=2}^{N} \left(\frac{n}{j^k}\right)^{1/r} < n^{1/r} \int_1^N t^{-k/r}\, dt$$

$$= \frac{r}{k-r} n^{1/r} \left(1 - N^{\frac{r-k}{r}}\right) \quad .$$

But $N \le n^{1/k}$, so

$$\sum_{j=2}^{N} \left(\frac{n}{j^k}\right)^{1/r} \leq \frac{r}{k-r} \cdot n^{1/r} \cdot \left(1 - n^{\frac{1}{k} \cdot \frac{r-k}{r}} \right)$$

(5.2.24) $$= \frac{r}{k-r} \left(n^{1/r} - n^{1/k}\right) .$$

We next investigate $\sum_{j=2}^{N} \left(\frac{n}{j^k}\right)^{1/2r}$

$$\sum_{j=2}^{N} \left(\frac{n}{j^k}\right)^{1/2r} < n^{1/2r} \int_1^N t^{-k/2r}\, dt$$

$$< n^{1/2r} \int_1^N t^{-1/2}\, dt$$

$$= 2 \cdot n^{1/2r} \left(N^{1/2} - 1\right)$$

(5.2.25) $$\leq 2\, n^{1/2r} \cdot n^{1/2k} - 2n^{1/2r} .$$

Substituting (5.2.25), (5.2.24), (5.2.23) and (5.2.22) into (5.2.20), we obtain

$$|E_{k,r}(n)| \leq 1 + \frac{n^{1/k}}{k-1} + \left(\frac{6}{\pi^2} + \frac{1}{r-1} + \frac{1}{n^{1/2r}} \right) n^{1/r}$$

$$+ 2n^{\frac{1}{2}\left(\frac{1}{r} + \frac{1}{k}\right)} - 2n^{1/2r} + \left(\frac{6}{\pi^2} + \frac{1}{r-1}\right)\frac{r}{k-r} \left(n^{1/r} - n^{1/k}\right)$$

$$< \left(\frac{6}{\pi^2} + \frac{1}{r-1}\right)\frac{k}{k-r} \cdot n^{1/r} + \frac{n^{1/k}}{k-1} + 2n^{\frac{1}{2}\left(\frac{1}{r} + \frac{1}{k}\right)}$$

which proves (5.2.7). To prove (5.2.8), we consider again the sum $\sum_{j=2}^{N} \left(\frac{n}{j^k}\right)^{1/r}$:

$$\sum_{j=2}^{N} \left(\frac{n}{j^k}\right)^{1/r} \le n^{1/r} \int_1^N t^{-k/r}\, dt$$

$$< n^{1/r} \int_1^N t^{-1}\, dt$$

$$< n^{1/r} \log N$$

$$\le n^{1/r} \log (n^{1/r})$$

$$= (1/r)\, n^{1/r} \log n \; . \tag{5.2.26}$$

Using (5.2.26) in place of (5.2.24) in the previous substitution, we obtain:

$$|E_{k,r}(n)| \le 1 + \frac{n^{1/k}}{k-1} + \left(\frac{6}{\pi^2} + \frac{1}{r-1} + \frac{1}{n^{1/2r}}\right) \cdot n^{1/r}$$

$$+ 2n^{\frac{1}{2}(\frac{1}{r} + \frac{1}{k})} - 2n^{1/2r} + \left(\frac{6}{\pi^2} + \frac{1}{r-1}\right) \frac{1}{r} \cdot n^{1/r} \cdot \log n$$

$$< \left(1/r \left(\frac{6}{\pi^2} + \frac{1}{r-1}\right) \cdot \log n + 4\right) \cdot n^{1/r} \; .$$

Q.E.D.

We state an immediate consequence of the two lemmas.

Corollary (5.2.27): If $2 \le r < k \le \infty$, and $n > t^r$, then the following four inequalities are true:

$$\left| \frac{\zeta(k)}{\zeta(r)} - \frac{q_{k,r}(n)}{n} \right| \le \frac{\left(\left(\frac{6}{\pi^2} + \frac{1}{r-1}\right) \log t\right) + 4}{t^{r-1}} \; . \tag{5.2.28}$$

$$(5.2.29) \qquad \left| \frac{\zeta(k)}{\zeta(r)} - \frac{q_{k,r}(n)}{n} \right| \leq \frac{\frac{k}{k-r}\left(\frac{6}{\pi^2} + \frac{1}{r-1}\right) + \frac{1}{k-1} + 2}{t^{r-1}}$$

$$(5.2.30) \qquad \left| c_{(k;r)} - \frac{q_{(k,r)}(n)}{n} \right| \leq$$

$$\leq \frac{(\gamma + \log t)\left(\frac{1}{\zeta(2)} + \frac{1}{r-1}\right) + 4 + \frac{\zeta(k)}{r-1} + \frac{\zeta(k)}{t}}{t^{r-1}}$$

$$(5.2.31) \qquad \left| c_{(k;r)} - \frac{q_{(k,r)}(n)}{n} \right| \leq$$

$$\leq \frac{k\left(\frac{1}{\zeta(2)(k-r)} + \frac{2}{2k-r} + \frac{1}{(k-r)(k-1)}\right) + 1 + \frac{\zeta(k)}{r-1} + \frac{\zeta(k)}{t}}{t^{r-1}} .$$

Proof: Set $n = t^r$ in (5.2.3), (5.2.4), (5.2.7) and (5.2.8), divide by n and the above results follow.

Q.E.D.

(§3) Bounds for $d_{(k;r)}$

In this section, we prove a group of lemmas, each of which assert that, given a certain lower bound for k, we have a certain upper bound for $d_{(k;r)}$. Throughout this section, we assume $r > 20$ (from Theorem (5.5.1)), and k and r are both integers.

Lemma (5.3.1). For $r > 20$, if

$$k > \left(1 + \frac{\log 1.5}{\log 2}\right)r - \frac{1}{\log 2}\left(\left(\frac{3}{5}\right)^r - 3 \cdot \left(\frac{2}{5}\right)^r\right) \tag{5.3.2}$$

then

$$d_{(k;r)} \leq 1 - \frac{1}{2^r} - \frac{1}{5^r+3^r+2^r} \quad . \tag{5.3.3}$$

Proof. First, let us assume $k > \left(1 + \frac{\log 1.5}{\log 2}\right)r$, i.e., $2^k > 3^r$. Then we have:

$$\left[\frac{n}{3^r}\right] \geq \left[\frac{n}{2^k}\right] . \tag{5.3.4}$$

Since $r > 20$, we have:

$$3^k > 3^r \cdot 3^{(\log 1.5/\log 2)r}$$

$$> 2 \cdot 5^r . \tag{5.3.5}$$

Let $s = \left[\frac{5^r}{2^r}\right] \cdot 2^r + 2^r < 5^r + 3^r + 2^r < 2 \cdot 5^r < 6^r$. By Lemma (4.2.3), we know

$$q_{(k;r)}(s) = \sum_{\substack{a^k b^r c \leq s \\ (a,b)=1 \\ a \in Q_2}} \mu(b)$$

$$= \sum_{\substack{(a,b)=1 \\ a\in Q_2}} \mu(b) \left[\frac{s}{a^k b^r}\right] .$$

From above, $a^k > s$ for $a \geq 3$, and $b^r > s$ for $b \geq 6$. Thus, from above, $[\frac{s}{a^k b^r}] = 0$ except when $a \leq 2$, $b \leq 5$ and one or both of $a = 1$, $b = 1$. We have

$$q_{(k;r)}(s) = [\frac{s}{1}] - [\frac{s}{2^r}] + [\frac{s}{2^k}] - [\frac{s}{3^r}] - [\frac{s}{5^r}] .$$

Since $5^r < s < 2\cdot 5^r$, $[\frac{s}{5^r}] = 1$. Thus

$$q_{(k;r)}(s) = s - [\frac{s}{2^r}] + [\frac{s}{2^k}] - [\frac{s}{3^r}] - 1$$

and since $2^r | s$, from (5.3.4) we have $q_{(k;r)}(s) \leq s - \frac{s}{2^r} - 1$.

Thus

$$d_{(k;r)} \leq \frac{q_{(k;r)}(s)}{s} \leq 1 - \frac{1}{2^r} - \frac{1}{s}$$

$$\leq 1 - \frac{1}{2^r} - \frac{1}{5^r+3^r+2^r} .$$

Now let k satisfy (5.3.2). We shall also assume $2^k < 3^r$. Again, (5.3.5) holds. Consider

$$3^r - 2^k \leq 3^r - 2^{\left(1 + \frac{\log 1.5}{\log 2}\right) r - \frac{1}{\log 2}\left((\frac{3}{5})^r - 3\cdot(\frac{2}{5})^r\right)}$$

$$= 3^r(1 - \exp\{-(\tfrac{3}{5})^r + 3\cdot(\tfrac{2}{5})^r\}) .$$

But it is trivial that $(1-e^{-x}) < x$ for $x > 0$. Thus

$$3^r-2^k < 3^r((\tfrac{3}{5})^r - 3(\tfrac{2}{5})^r)$$

$$= (\tfrac{9}{5})^r - 3\cdot(\tfrac{6}{5})^r$$

(5.3.6) $$< \frac{3^r-2^r}{(\frac{5}{3})^r+2} \quad .$$

Let $s_1 = [\frac{5^r}{3^r}] \cdot 3^r+3^r$ and $s_2 = [\frac{s_1}{2^r}] \cdot 2^r + 2^r$. We claim that:

(5.3.7) $$[\ \frac{s_1}{3^r}\ ] = [\ \frac{s_1}{2^k}\ ]$$

and

(5.3.8) $$2^k \nmid s_2 \quad .$$

(5.3.7) obviously follows from (5.3.6) and the definition of s_1. Let us assume that (5.3.8) is not satisfied. Let $m = [\ \frac{s_1}{3^r}\ ] = (\ \frac{s_1}{3^r}\ )$. We have, since $2^k|s_2$:

$$(m+1) \cdot 2^k < s_1 + 2^r$$

implies

$$2^k + m \cdot 2^k < s_1 + 2^r$$

implies

$$2^k + m \cdot 2^k < m \cdot 3^r + 2^r$$

implies

$$2^k - 2^r < m(3^r - 2^k) \quad .$$

But $m < (\frac{5}{3})^r + 1$, so

$$(5.3.9) \qquad \frac{(2^k - 2^r)}{(\frac{5}{3})^r + 1} < (3^r - 2^k) \quad .$$

A straight forward calculation shows that (5.3.9) contradicts (5.3.6), and this establishes (5.3.8)

But (5.3.7) and (5.3.8) together show $[\frac{s_2}{3^r}] = [\frac{s_2}{2^k}]$.
Thus, arguing as before, we have:

$$q_{(k;r)}(s_2) = s_2 - [\frac{s_2}{2^r}] + [\frac{s_2}{2^k}] - [\frac{s_2}{3^r}] - [\frac{s_2}{5^r}]$$

$$= s_2 - \frac{s_2}{2^r} - 1 \ .$$

Thus, as above, for $s = s_2$

$$\frac{q_{(k;r)}(s)}{s} < 1 - \frac{1}{2^r} - \frac{1}{5^r+3^r+2^r}$$

and

$$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{1}{5^r+3^r+2^r} \quad .$$

Q.E.D.

<u>Lemma (5.3.10)</u>. For $r > 20$, if

$$(5.3.11) \qquad k > (1 + \frac{\log 1.5}{\log 2})r + \frac{1}{2} \cdot \frac{1}{\log 2} \cdot (\frac{2}{3})^{r(1+(\log 1.5/\log 2))}$$

then

$$(5.3.12) \qquad d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{2}{5^r+3^r+2^r} \quad .$$

<u>Proof</u>. Let us assume (5.3.11) and consider 2^k-3^r:

$$2^k - 3^r \geq 2^r \cdot 1.5^r \cdot 2^{\frac{1}{2 \log 2}(\frac{2}{3})^{r(1+(\log 1.5/\log 2))}} - 3^r$$

$$= 3^r(\exp\ (\tfrac{1}{2}\ (\tfrac{2}{3})^{r(1+(\log 1.5/\log 2))})-1)\ .$$

Since $(e^x - 1) > x$ for $x > 0$,

$$2^k - 3^r > 3^r \cdot \frac{1}{2} \cdot (\frac{2}{3})^{r(1+(\log 1.5/\log 2))}$$

$$= \frac{1}{2} \cdot 2^r\ (\frac{2}{3})^{r \cdot (\log 1.5/\log 2)}\ .$$

Multiplying by $(\frac{5}{3})^r$ and recalling $r > 20$, we obtain:

$$(\frac{5}{3})^r(2^k - 3^r) > 2^r \cdot \frac{1}{2} \cdot (\frac{2}{3})^{r \cdot (\log 1.5/\log 2)} \cdot (\frac{5}{3})^r$$

(5.3.13) $$> 2^r\ .$$

Define s_1, s_2 and m as in the previous lemma. We have, by (5.3.13),

$$m(2^k - 3^r) > 2^r$$

implies

$$m \cdot 2^k > m \cdot 3^r + 2^r = s_1 + 2^r > s_2\ .$$

Thus $[\ \frac{s_2}{2^k}\ ] \leq m-1$ and

(5.3.14) $$[\ \frac{s_2}{3^r}\ ] \geq [\ \frac{s_2}{2^k}\ ] + 1\ .$$

Repeating the argument from the previous lemma, but using (5.3.14), we have:

$$q_{(k;r)}(s_2) \leq s_2 - \left(\frac{s_2}{2^r}\right) - 2$$

which implies

$$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{2}{5^r+3^r+2^r} \quad .$$

Q.E.D.

Lemma (5.3.15). For $r > 20$, if

(5.3.16) $$k > \left(1 + \frac{\log 1.5}{\log 2}\right)r + \frac{1}{\log 2}\left(\frac{3}{5}\right)^r$$

then

(5.3.17) $$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{3^r}{10^r+9^r+6^r} \quad .$$

Proof. Let us examine (2^k-3^r):

$$\begin{aligned}(2^k-3^r) &\geq 2^r \cdot 1.5^r \cdot 2^{(\frac{3}{5})^r/\log 2} - 3^r \\ &= 3^r(\exp\{(\tfrac{3}{5})^r\} - 1) \\ &> 3^r \cdot (\tfrac{3}{5})^r \\ &= (\tfrac{9}{5})^r \quad .\end{aligned}$$

Thus, for $m \geq (\frac{10}{9})^r$, we have

$$m \cdot (2^k-3^r) > (\tfrac{9}{5})^r \cdot (\tfrac{10}{9})^r$$

(5.3.18) $$= 2^r \quad .$$

Define $s_1 = ([(\frac{10}{9})^r]+1)\cdot 3^r$ and $s_2 = ([\frac{s_1}{2^r}]+1)\cdot 2^r$, then, arguing

as in the previous lemma, we have from (5.3.18) that

(5.3.19) $$[\frac{s_2}{2^k}] \leq [\frac{s_2}{3^r}] - 1 .$$

Since $r > 20$, $s_2 < 5^r$, and we have

$$q_{(k;r)}(s_2) = [\frac{s_2}{1}] - [\frac{s_2}{2^r}] + [\frac{s_2}{2^k}] - [\frac{s_2}{3^r}] - [\frac{s_2}{5^r}]$$

(5.3.20) $$\leq s_2 - \frac{s_2}{2^r} - 1$$

by (5.3.19). Since $s_2 < (\frac{10}{9})^r \cdot 3^r + 3^r + 2^r = (\frac{10}{3})^r + 3^r + 2^r$ we have by (5.3.20)

$$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{1}{(\frac{10}{3})^r + 3^r + 2^r}$$

$$= 1 - \frac{1}{2^r} - \frac{3^r}{10^r + 9^r + 6^r} .$$

Q.E.D.

Lemma (5.3.21). For $r > 20$, if

(5.3.22) $$k \geq (1 + \frac{\log 1.5}{\log 2})r + \frac{1}{2 \log 2} \cdot (\frac{9}{10})^r$$

then

(5.3.23) $$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{1}{3^r+2^r} .$$

Proof. Again, we examine (2^k-3^r):

$$2^k-3^r \geq 2^r \cdot 1.5^r \cdot 2^{\frac{1}{2 \log 2}(\frac{9}{10})^r} - 3^r$$

$$= 3^r (\exp \{\frac{1}{2}(\frac{9}{10})^r\} - 1)$$

$$> \frac{1}{2} \cdot 3^r \left(\frac{9}{10}\right)^r$$

(5.3.24) $$> 2^r$$

since $r > 20$. Let $s = \left(\left[\frac{3^r}{2^r}\right]+1\right) \cdot 2^r$. Then, on the strength of (5.3.24), we have $\left[\frac{s}{2^k}\right] = 0$. Thus we have

$$q_{(k;r)}(s) = s - \left[\frac{s}{2^r}\right] - 1$$

and since $s < 3^r + 2^r$, and $2^r | s$,

$$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{1}{3^r+2^r} .$$

Q.E.D.

<u>Lemma (5.3.25).</u> For $r > 20$, if

(5.3.26) $$k \geq 2\left(\frac{\log 3}{\log 2} - \frac{1}{2}\right) r - \frac{2}{\log 2} \left(\frac{9}{10}\right)^r$$

then

(5.3.27) $$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{4^r}{3^r(4^r+2^r)} .$$

<u>Proof.</u> For $r > 20$, if k satisfies (5.3.26), then $2^k > 4^r+3^r+2^r$. Let $s_1 = \left(\left[\frac{4^r}{3^r}\right] + 1\right) \cdot 3^r$ and $s_2 = \left(\left[\frac{s_1}{2^r}\right] + 1\right) \cdot 2^r$ and arguing as in the previous lemma, we find

$$q_{(k;r)}(s_2) = 1 - \frac{s_2}{2^r} - \frac{s_1}{3^r} + \left[\frac{s_2}{2^k}\right] - \left[\frac{s_2}{5^r}\right]$$

$$\frac{q_{(k;r)}(s_2)}{s_2} = 1 - \frac{1}{2^r} - \frac{s_1}{3^r \cdot s_2} + 0$$

$$< 1 - \frac{1}{2^r} - \frac{4^r}{3^r(4^r+2^r)} \quad .$$

Thus $d_{(k;r)}$ satisfies (5.3.27).

Q.E.D.

Lemma (5.3.28). For $r > 20$, if

(5.3.29) $$k \geq \left(\frac{\log 5}{\log 2}\right) r - \left(\frac{5}{6}\right)^r / \log 2$$

then

(5.3.30) $$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{5^r}{3^r(5^r+2 \cdot 2^r)} \quad .$$

Proof. Since $r > 20$ and k satisfies (5.3.29), $2^k > \frac{1}{2} \cdot 5^r+3^r+2^r$.

Set $s_1 = \left(\left[\frac{\frac{1}{2} \cdot 5^r}{3^r} \right] + 1\right) \cdot 3^r$ and $s_2 = \left(\left[\frac{s_1}{2^r} \right] + 1\right) \cdot 2^r$.

Then as before:

$$\frac{q_{(k;r)}(s_2)}{s_2} < 1 - \frac{1}{2^r} - \frac{\frac{1}{2} \cdot 5^r}{3^r(\frac{1}{2} \cdot 5^r+2^r)}$$

$$= 1 - \frac{1}{2^r} - \frac{5^r}{3^r(5^r+2 \cdot 2^r)} \quad .$$

and $d_{(k;r)}$ satifies (5.3.30).

Q.E.D.

Lemma (5.3.31). For $r > 20$, if

(5.3.32) $$k > \left(\frac{\log 5}{\log 2}\right) r + \frac{1}{2}\left(\frac{5}{6}\right)^r / \log 2$$

then

(5.3.33) $$d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{5^r + 2\cdot 3^r - \left(\frac{6}{5}\right)^r}{3^r\cdot(5^r+3^r+2^r)} \quad .$$

<u>Proof</u>. Let us examine $2^k - 5^r$:

$$2^k - 5^r > 2^{\left(\frac{\log 5}{\log 2}\right) r} \cdot 2^{\frac{1}{2}\left(\frac{5}{6}\right)^r / \log 2} - 5^r$$

$$= 5^r\left(\exp\left\{\frac{1}{2}\left(\frac{5}{6}\right)^r\right\} - 1\right)$$

$$> \frac{1}{2} \cdot 5^r \left(\frac{5}{6}\right)^r$$

$$> 3^r + 2^r$$

since $r > 20$. Applying the same arguments as before, we set

$s_1 = \left(\left[\dfrac{5^r}{3^r}\right] + 1\right) \cdot 3^r$, $s_2 = \left(\left[\dfrac{s_1}{2^r}\right] + 1\right) \cdot 2^r$ and find:

$$\left[\frac{s_2}{2^r}\right] = \frac{s_2}{2^r}$$

$$\left[\frac{s_2}{3^r}\right] > \frac{5^r}{(5^r+2^r)\cdot 3^r}$$

$$\left[\frac{s_2}{2^k}\right] = 0$$

$$\left[\frac{s_2}{5^r}\right] = 1$$

$$s_2 < 5^r + 3^r + 2^r \quad .$$

Thus

$$d_{(k;r)} \leq \frac{q_{(k;r)}(s_2)}{s_2}$$

$$< 1 - \frac{1}{2^r} - \frac{1}{5^r+3^r+2^r} - \frac{5^r}{3^r(5^r+2^r)}$$

$$< 1 - \frac{1}{2^r} - \frac{5^r+2\cdot 3^r-(\frac{6}{5})^r}{3^r(5^r+3^r+2^r)} \quad .$$

Q.E.D.

(§4) Some Algebraic Results

In this section, a number of similar lemmas of a technical nature are proved. Throughout this section, we define $a = r$ and $b = (k-r)$. (We assume $a > 20$ from Theorem (5.5.1).)

Lemma (5.4.1). For $a > 20$, if

$$b \leq a\frac{\log 1.5}{\log 2} - \frac{(\frac{3}{5})^a - 3(\frac{2}{5})^a}{\log 2} \quad , \tag{5.4.2}$$

then

$$60^a 30^b + 3\cdot 16^a \cdot 60^b \geq 24^a \cdot 60^b + 40^a 60^b \quad . \tag{5.4.3}$$

Proof. Consider

$$60^a 30^b - (40^a + 24^a - 3\cdot 16^a)60^b = f(a,b) \quad . \tag{5.4.4}$$

Clearly, if $f(b) \leq 0$, $\frac{\partial f}{\partial b} < 0$. Let $f(a,b_o) = 0$ (assuming $b_o = b_o(a)$). Then for $b < b_o$, (5.4.3) is satisfied. Thus we wish to show

$$b_o \geq a\frac{\log 1.5}{\log 2} - \frac{(\frac{3}{5})^a - 3\cdot(\frac{2}{5})^a}{\log 2}$$

or

$$s < (\tfrac{3}{5})^a - 3(\tfrac{2}{5})^a \tag{5.4.5}$$

where s is defined by:

$$b_o = a\frac{\log 1.5}{\log 2} - \frac{s}{\log 2} \quad . \tag{5.4.6}$$

Combining (5.4.6) with (5.4.4), we find:

$$60^a 30^{\left(a \frac{\log 1.5}{\log 2} - \frac{s}{\log 2}\right)} = (40^a+24^a-3\cdot 16^a)60^{\left(a \frac{\log 1.5}{\log 2} - \frac{s}{\log 2}\right)}$$

implies

$$(\tfrac{3}{2})^a = (1+(\tfrac{24}{40})^a-3(\tfrac{16}{40})^a)\cdot 2^{\left(a \frac{\log 1.5}{\log 2} - \frac{s}{\log 2}\right)}$$

$$= (1+(\tfrac{3}{5})^a-3(\tfrac{2}{5})^a)(\tfrac{3}{2})^a\cdot e^{-s}$$

implies

$$(1-e^{-s}) = ((\tfrac{3}{5})^a-3\cdot(\tfrac{2}{5})^a)e^{-s}$$

implies

$$(e^s-1) = (\tfrac{3}{5})^a-3\cdot(\tfrac{2}{5})^a$$

implies

$$s < (\tfrac{3}{5})^a-3(\tfrac{2}{5})^a .$$

Q.E.D.

Lemma (5.4.7). For $a \geq 91$, if

(5.4.8) $$b \leq a(\frac{\log 1.5}{\log 2}) + \frac{\frac{1}{2}(\frac{2}{3})^{a(1+(\log 1.5/\log 2))}}{\log 2}$$

then

(5.4.9) $$300^a 30^b-200^a 60^b > -\frac{200^a\cdot 20^b}{1.99} .$$

Proof. Arguing as before, let b_o be the point of equality and let

(5.4.10) $$b_o = a(\frac{\log 1.5}{\log 2}) + \frac{s}{\log 2} .$$

We wish to show $s > \frac{1}{2}(\frac{2}{3})^{a(1+(\log 1.5/\log 2))}$. By (5.4.9) and (5.4.10), we have:

$$300^a \cdot 30^{(a(\log 1.5/\log 2) + s/\log 2)} - 200^a 60^{(a(\log 1.5/\log 2) + s/\log 2)}$$

$$= 200^a 20^{(a(\log 1.5/\log 2) + s/\log 2)} \cdot (\frac{-1}{1.99})$$

implies

$$(\frac{3}{2})^a (\frac{3}{2})^{(a(\log 1.5/\log 2) + s/\log 2)} - 3^{(a(\log 1.5/\log 2) + s/\log 2)}$$

$$= -\frac{1}{1.99}$$

implies

$$(\frac{3}{2})^{(a(1+(\log 1.5/\log 2))+ s/\log 2)}(1-e^s) = -\frac{1}{1.99}$$

implies

(5.4.11) $$(e^s-1) = \frac{1}{1.99}(\frac{2}{3})^{(a(1+(\log 1.5/\log 2))+ s/\log 2)} .$$

Since $a \geq 91$, (5.4.11) implies

$$s > \frac{1}{2}(\frac{2}{3})^{a(1+(\log 1.5/\log 2))} .$$

Q.E.D.

Lemma (5.4.12). For $a > 20$, if

(5.4.13) $$b \leq \left(\frac{\log 1.5}{\log 2}\right)a + \left(\frac{3}{5}\right)^a/\log 2$$

then

(5.4.14) $$300^a 30^b - 200^a 60^b \geq - 120^a 60^b \quad .$$

Proof. Defining b_o as before and s by:

(5.4.15) $$b_o = \left(\frac{\log 1.5}{\log 2}\right)a + s/\log 2 \quad .$$

We wish to show that $s > \left(\frac{3}{5}\right)^a$. But from (5.4.14) and (5.4.15) we have

$$300^a 30^{(a(\log 1.5/\log 2)+s/\log 2)} - 200^a 60^{(a(\log 1.5/\log 2)+s/\log 2)}$$
$$= -120^a 60^{(a(\log 1.5/\log 2)+s/\log 2)}$$

implies

$$\left(\frac{5}{2}\right)^a \left(\frac{1}{2}\right)^{(a(\log 1.5/\log 2)+s/\log 2)} - \left(\frac{5}{3}\right)^a = -1$$

implies

$$\left(\frac{5}{3}\right)^a (e^{-s} - 1) = -1$$

implies

$$1 - e^{-s} = \left(\frac{3}{5}\right)^a$$

implies

$$s > (\tfrac{3}{5})^a .$$

Q.E.D.

Lemma (5.4.16). For $a > 20$, if

(5.4.17) $$b \le (\frac{\log 1.5}{\log 2}) a + (\frac{9}{10})^a/2 \log 2$$

then

(5.4.18) $$600^a 30^b - 400^a 60^b > -\frac{1}{2}(360^a 60^b)$$

and

(5.4.19) $$540^a 30^b - 360^a 60^b > -\frac{1}{2}(324^a 60^b) .$$

Proof. Notice (5.4.18) and (5.4.19) are equivalent. We shall prove (5.4.18). Select b_o as before and define s by:

(5.4.20) $$b_o = (\frac{\log 1.5}{\log 2}) a + \frac{s}{\log 2} .$$

We wish to show $s > (\frac{9}{10})^a/2$. Combining (5.4.20) and (5.4.18), we find:

$$600^a 30^{(a(\log 1.5/\log 2)+s/\log 2)} - 400^a 60^{(a(\log 1.5/\log 2)+s/\log 2)}$$

$$= -\frac{1}{2} \cdot 360^a \cdot 60^{(a(\log 1.5/\log 2)+s/\log 2)}$$

implies

$$(\tfrac{5}{3})^a (\tfrac{1}{2})^{(a(\log 1.5/\log 2)+s/\log 2)} - (\tfrac{10}{9})^a \ge -\frac{1}{2}$$

implies

$$\left(\frac{10}{9}\right)^a (e^{-s}-1) = -\frac{1}{2}$$

implies

$$(1-e^{-s}) = \frac{1}{2}\left(\frac{9}{10}\right)^a$$

implies

$$s > \frac{1}{2}\left(\frac{9}{10}\right)^a .$$

Q.E.D.

Lemma (5.4.21). For $a > 20$, if

(5.4.22) $$b \leq 2\left(\frac{\log 3}{\log 2} - 1\right)a - 2\left(\frac{9}{10}\right)^a/\log 2$$

then

(5.4.23) $$180^a 30^b - 80^a 60^b \geq 2\cdot 72^a 60^b .$$

Proof. Again, define s by

(5.4.24) $$b_o = 2\left(\frac{\log 3}{\log 2} - 1\right)a - \frac{s}{\log 2} .$$

We wish to show $s < 2 \cdot \left(\frac{9}{10}\right)^a$. From (5.4.23) and (5.4.24) we have:

$$180^a 30^{\left(a\cdot 2\left(\frac{\log 3}{\log 2} -1\right) - \frac{s}{\log 2}\right)} - 80^a 60^{\left(a\cdot 2\left(\frac{\log 3}{\log 2} - 1\right) - \frac{s}{\log 2}\right)}$$

$$= 2\cdot 72^a 60^{\left(a\cdot 2\left(\frac{\log 3}{\log 2} -1\right) - \frac{s}{\log 2}\right)}$$

implies

$$(\tfrac{9}{4})^{a}-2^{(a\cdot 2(\frac{\log 3}{\log 2}-1)-\frac{s}{\log 2})} = 2\cdot(\tfrac{9}{10})^{a}\cdot 2^{(a\cdot 2(\frac{\log 3}{\log 2}-1)-\frac{s}{\log 2})}$$

implies

$$(\tfrac{9}{4})^{a}(1-e^{-s}) = 2(\tfrac{9}{10})^{a}\cdot(\tfrac{9}{4})^{a}\ e^{-s}$$

implies

$$(e^{s}-1) = 2\cdot(\tfrac{9}{10})^{a}$$

implies

$$s < 2\cdot(\tfrac{9}{10})^{a} \quad .$$

Q.E.D.

Lemma (5.4.25). For $a > 20$, if

(5.4.26) $$b \leq (\frac{\log 5}{\log 2}-1)a - (\tfrac{5}{6})^{a}/\log 2$$

then

(5.4.27) $$5040^{a}3^{b}-2016^{a}6^{b} \geq 1680^{a}\cdot 6^{b} \quad .$$

Proof. Define s by

(5.4.28) $$b_o = a(\frac{\log 5}{\log 2}-1) - s/\log 2 \quad .$$

We wish to show $s < (\frac{5}{6})^{a}$. From (5.4.27) and (5.4.28) we have:

$$5040^{a}3^{(a(\frac{\log 5}{\log 2} - 1)-s/\log 2)} - 2016^{a}6^{(a(\frac{\log 5}{\log 2} - 1)-s/\log 2)}$$

$$= 1680^{a}6^{(a(\frac{\log 5}{\log 2} -1)-s/\log 2)}$$

implies

$$(\frac{5}{2})^{a} - (\frac{5}{2})^{a} e^{-s} = (\frac{5}{6})^{a} (\frac{5}{2})^{a} e^{-s}$$

implies

$$(e^{s}-1) = (\frac{5}{6})^{a}$$

implies

$$s < (\frac{5}{6})^{a} .$$

Q.E.D.

Lemma (5.4.29). For $a > 20$, if

(5.4.30) $$b \leq (\frac{\log 5}{\log 2} - 1)a + \frac{1}{2}(\frac{5}{6})^{a}/\log 2$$

then

(5.4.31) $$6300^{a}3^{b} - 2520^{a}6^{b} \geq -\frac{1}{2} \cdot 2100^{a}6^{b} .$$

Proof. Define s by

(5.4.32) $$b_o = (\frac{\log 5}{\log 2} - 1)a + s/\log 2 .$$

We wish to show $s > \frac{1}{2}(\frac{5}{6})^{a}$. From (5.4.31) and (5.4.32) we have:

$$6300^{a}3^{(a(\frac{\log 5}{\log 2} - 1)+s/\log 2)} - 2520^{a}6^{(a(\frac{\log 5}{\log 2} -1)+s/\log 2)}$$

$$= -\frac{1}{2} \cdot 2100^{a} \cdot 6^{(a(\frac{\log 5}{\log 2} - 1)+s/\log 2)}$$

implies

$$(\frac{5}{2})^{a} - (\frac{5}{2})^{a} e^{s} = -\frac{1}{2} (\frac{5}{6})^{a} (\frac{5}{2})^{a} e^{s}$$

implies

$$(1-e^{s}) = -\frac{1}{2}(\frac{5}{6})^{a} e^{s}$$

implies

$$(1-e^{-s}) = \frac{1}{2} (\frac{5}{6})^{a}$$

implies

$$s > \frac{1}{2} (\frac{5}{6})^{a} \quad .$$

Q.E.D.

(§5) The Main Results

We can now proceed to prove our main results. Again, we assume $r > 20$ throughout and set $a = r$, $b = (k-r)$.

Theorem (5.5.1). For $20 < r < k \leq \infty$, we have:

$$d_{k,r} \leq d_{(k;r)} = \frac{q_{(k;r)}(\hat{n}_o)}{\hat{n}_o} < D_{k,r} < D_{(k;r)} \tag{5.5.2}$$

for some $2^r \leq \hat{n}_o < 8^r$. Also $d_{k,r} = \dfrac{q_{k,r}(n_o)}{n_o}$ for some $2^r \leq n_o < 8^r$.

Proof: It can easily be seen that

$$D_{k,r} = \frac{\zeta(k)}{\zeta(r)} = \prod_p \frac{(1 - \frac{1}{p^r})}{(1 - \frac{1}{p^k})} < \prod_p (1 - \frac{1}{p^r} + \frac{1}{p^k}) = D_{(k;r)} .$$

From the definition of the (k,r) integers, it can be deduced that n is a (k,r) integer if, in its canonical expansion $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j}$, none of the α_i lie in $\bigcup_{\ell=o}^{\infty} [r+k\cdot\ell, k\cdot(\ell+1))$. Thus $Q_{k,r} \subset Q_{(k;r)}$ and $q_{k,r}(x) \leq q_{(k;r)}(x)$. Thus $d_{k,r} \leq d_{(k;r)}$.

From (5.5.2), if $D_{k,r} - \dfrac{q_{(k;r)}(n)}{n} > C$, since $d_{(k;r)} \leq \dfrac{q_{(k;r)}(n)}{n}$, we have both $D_{k,r} - d_{k,r} > C$, and $D_{(k;r)} - d_{(k;r)} > C$. If we set

$$f(t) = t^{1-r} \{((\frac{6}{\pi^2} + \frac{1}{r-1}) \log t) + 4\} \tag{5.5.3}$$

and

$$(5.5.4) \qquad \hat{f}(t) = t^{1-r}\{(\gamma + \log t)(\frac{1}{\zeta(2)} + \frac{1}{r-1}) + 4 + \frac{\zeta(r)}{r-1} + \frac{\zeta(r)}{t}\}$$

then from Corollary (5.2.27), if $f(t) < C$, then $n_o < t^r$, and if $\hat{f}(t) < C$, then $\hat{n}_o < t^r$. Thus to prove the theorem, we shall find a value of n (which will always be in the interval $[2^r, 6^r)$ at which we can show that

$$D_{k,r} - \frac{q_{(k;r)}(n)}{n} > C$$

where $C = C(k,r)$ is a number so large that $f(8) < C$ and $\hat{f}(8) < C$ (in many cases this can be improved to $f(6) < C$ and $\hat{f}(6) < C$). That $n_o \geq 2^r$ and $\hat{n}_o \geq 2^r$ is obvious.

We shall therefore proceed to find these values of C. We consider a number of cases separately, determining C for each case.

<u>Case 1</u>

$$r < k \leq (1 + \frac{\log 1.5}{\log 2})r - \frac{(\frac{3}{5})^r - 3\cdot(\frac{2}{5})^r}{\log 2} .$$

Set $r = a$ and $(k-r)=b$. We have

$$(5.5.5) \qquad D_{k,r} = \frac{\zeta(k)}{\zeta(r)} = \frac{\zeta(a+b)}{\zeta(a)} > \frac{1 + \frac{1}{2^{a+b}} + \frac{1}{3^{a+b}}}{1 + \frac{1}{2^a} + \frac{1}{3^a} + \frac{1}{4^a} + \frac{1}{5^a} + (\frac{a+5}{a-1})\cdot\frac{1}{6^a}}$$

$\frac{q_{(k;r)}(2^a)}{2^a} = 1 - \frac{1}{2^a}$, and thus $d_{(k;r)} \leq 1 - \frac{1}{2^a}$. Thus

$$D_{k,r} - d_{(k;r)} \geq \frac{\zeta(a+b)}{\zeta(a)} - (1 - \frac{1}{2^a})$$

(5.5.6) $$\geq \frac{60^a30^b+40^a20^b-40^a60^b-24^a60^b-20^a60^b(6/(a-1))}{60^b \cdot H}$$

where

$$H = (120^a+60^a+40^a+30^a+24^a+(\frac{a+5}{a-1})20^a) \quad .$$

Since $k \leq r(1 + \frac{\log 1.5}{\log 2}) - \frac{(\frac{3}{5})^r - 3(\frac{2}{5})^r}{\log 2}$, we can apply Lemma (5.4.1) to find

(5.5.7) $$60^a30^b+3\cdot 16^a60^b \geq 24^a60^b+40^a60^b \quad .$$

Combining (5.5.6) and (5.5.7), we find

(5.5.8) $$D_{k,r} - d_{(k;r)} \geq \frac{40^a20^b - ((\frac{6}{a-1})20^a+3\cdot 16^a)\cdot 60^b}{60^b \cdot H}$$

$$\geq \frac{1}{3^a3^b} - \frac{6}{(a-1)6^a} - \frac{1}{(7.5)^a} \quad .$$

Since $b < a \cdot (\frac{\log 1.5}{\log 2})$, we obtain:

$$D_{k,r} - d_{(k;r)} > \frac{1}{(5.705)^a} - \frac{6}{(a-1)6^a} - \frac{1}{(7.5)^a}$$

$$= C, \text{ say.}$$

From (5.5.3) and (5.5.4), for $a > 20$, $f(8) < C$ and $\hat{f}(8) < C$. Thus $d_{k,r} = \frac{q_{k,r}(n_o)}{n_o}$ and $d_{(k;r)} = \frac{q_{(k;r)}(\hat{n}_o)}{\hat{n}_o}$ for

$2^a \leq n_o < 8^a$ and $2^a \leq \hat{n}_o < 8^a$. This completes Case 1.

Case II

$$(1+\frac{\log 1.5}{\log 2})r - \frac{(\frac{3}{5})^r - 3\cdot(\frac{2}{5})^r}{\log 2} < k \leq (1+\frac{\log 1.5}{\log 2})r +$$

$$+\frac{1}{2}(\frac{2}{3})^{r(1+\log 1.5/\log 2)}/\log 2$$

By Lemma (5.3.1), $d_{(k;r)} \leq 1 - \frac{1}{2^r} - \frac{1}{5^r+3^r+2^r}$. Combining this with (5.5.5), we find:

$$D_{k,r} - d_{(k;r)} \geq \frac{120^a 60^b+60^a 30^b+40^a 20^b}{60^b\cdot H} - (1-\frac{1}{2^a}-\frac{1}{5^a+3^a+2^a}) \tag{5.5.9}$$

$$\geq \{300^a\cdot 30^b+180^a 30^b+120^a 30^b+200^a 20^b+120^a 20^b$$

$$+80^a 20^b-200^a 60^b-120^a 60^b-80^a 60^b-72^a 60^b-48^a 60^b$$

$$-100^a 60^b(\frac{a+5}{a-1})-60^a\cdot 60^b(\frac{a+5}{a-1})-40^a 60^b(\frac{a+5}{a-1})+2\cdot 60^a 60^b$$

$$+40^a 60^b+100^a 60^b\}/(5^a+3^a+2^a)\cdot 60^b\cdot H \quad .$$

By Lemma (5.4.7) if $a \geq 91$, $300^a 30^b-200^a 60^b > \frac{-200^a\cdot 20^b}{1.99}$. Combining this with (5.5.9), we find,

$$D_{k,r}-d_{(k;r)} \geq \{(1-\frac{1}{1.99})(200^a 20^b)-100^a 60^b(\frac{6}{a-1}) +$$

$$+ (1-\frac{1}{1.99})(120^a 20^b)-60^b(72^a+48^a+80^a$$

$$+(60^a+40^a)(\frac{6}{a-1}))\}/(5^a+3^a+2^a)\cdot 60^b\cdot H \,. \tag{5.5.10}$$

Let us assume $a > 90$. Then from (5.5.10) and

$b \le (\frac{\log 1.5}{\log 2})a + \frac{1}{2}(\frac{2}{3})^{r(1+\frac{\log 1.5}{\log 2})}/\log 2$, we deduce:

$$D_{k,r} - d_{(k;r)} \ge .48 \frac{(200)^a \cdot 20^b}{5^a \cdot 120^a \cdot 60^b}$$

$$\ge .48 (\frac{1}{5.705})^a = C.$$

But $\hat{f}(6) < C$ and $f(6) < C$, so $n_o < 6^a$ and $\hat{n}_o < 6^a$. If $20 < a \le 90$, Case II is satisfied vacuously, since there are no k satisfying Case II for $20 < a \le 90$. This concludes Case II.

<u>Case III</u>

$$(1 + \frac{\log 1.5}{\log 2})r + \frac{1}{2}(\frac{2}{3})^{r(1+\frac{\log 1.5}{\log 2})}/\log 2 < k$$

$$\le (1 + \frac{\log 1.5}{\log 2})r + (\frac{3}{5})^r/\log 2 .$$

By Lemma (5.3.10), $d_{(k;r)} \le 1 - \frac{1}{2^r} - \frac{2}{5^r+3^r+2^r}$.

Combining this with (5.5.5), we find

$$\begin{aligned} D_{k,r}-d_{(k;r)} \ge \{&300^a30^b+180^a30^b+120^a30^b+200^a20^b+120^a20^b \\ &+80^a20^b-200^a60^b-80^a60^b-72^a60^b-48^a60^b-100^a60^b(\tfrac{a+5}{a-1}) \\ &-60^a60^b(\tfrac{a+5}{a-1})-40^a60^b(\tfrac{a+5}{a-1})+100^a60^b+75^a60^b+3\cdot 60^a60^b \\ &+2\cdot 40^a60^b\}/(5^a+3^a+2^a)\cdot 60^b\cdot H . \end{aligned} \tag{5.5.11}$$

By Lemma (5.4.12), $300^a 30^b - 200^a 60^b \geq -120^a 60^b$. Combining this with (5.5.11), we find:

$$D_{k,r} - d_{(k;r)} \geq \{200^a 20^b - 100^a 60^b (\frac{6}{a-1})$$

$$-60^b (72^a + 80^a)\} / (5^a + 3^a + 2^a) \cdot 60^b \cdot H .$$

Again, for $a > 90$, $b \leq (\frac{\log 1.5}{\log 2}) r + (\frac{3}{5})^r / \log 2$, we have

$$D_{k,r} - d_{(k;r)} \geq .48 \frac{(200)^a \cdot 20^b}{5^a \cdot 120^a \cdot 60^b}$$

$$\geq .48 (\frac{1}{5.705})^a = C .$$

But $\hat{f}(6) < C$ and $f(6) < C$, so $n_o < 6^a$ and $\hat{n}_o < 6^a$. If $20 < a \leq 90$, Case III is satisfied vacuously. This completes Case III.

Case IV

$$(1 + \frac{\log 1.5}{\log 2}) r + (\frac{3}{5})^r / \log 2 < k \leq (1 + \frac{\log 1.5}{\log 2}) r$$

$$+ (\frac{9}{10})^r / 2 \log 2 .$$

By Lemma (5.3.15), $d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{3^r}{10^r + 9^r + 6^r}$.

Combining this with (5.5.5), we find:

$$(5.5.12) \qquad D_{k,r}-d_{(k;r)} \geq \{(600^a+540^a+360^a)30^b+(400^a+360^a+240^a)20^b$$

$$+ (-400^a-2\cdot 240^a-216^a-144^a+200^a+180^a+120^a+90^a+72^a$$

$$- (+200^a+180^a+120^a)(\frac{a+5}{2-1}))60^b\}/(10^a+9^a+6^a)60^b\cdot H \ .$$

By Lemma (5.4.16), $600^a30^b-400^a60^b > -\frac{1}{2}(360^a60^b)$ combining this with (5.5.12), we find:

$$D_{k,r}-d_{(k;r)} \geq \{\frac{1}{2}\cdot 360^a\cdot 60^b-(\frac{1}{2}\cdot 324^a+2\cdot 240^a+216^a+144^a +$$

$$+ (200^a+180^a+120^a)(\frac{6}{a-1}))60^b\}/(10^a+9^a+6^a)60^b\cdot H \ .$$

Since $a > 20$, we have:

$$D_{k,r}-d_{(k;r)} > \frac{\frac{1}{3}\cdot 360^a\cdot 60^b}{10^a\cdot 60^b\cdot 120^a} = \frac{1}{3}\cdot (\frac{3}{10})^a = C \ .$$

Again $\hat{f}(6) < C$ and $f(6) < C$, so $n_o < 6^a$ and $\hat{n}_o < 6^a$. This completes Case IV.

Case V

$$(1 + \frac{\log 1.5}{\log 2})r + \frac{1}{2}(\frac{9}{10})^r/\log 2 < k \leq 2(\frac{\log 3}{\log 2} - \frac{1}{2})r-2(\frac{9}{10})^r/\log 2 \ .$$

By Lemma (5.3.21), $d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{1}{3^r+2^r}$. Combining this with (5.5.5), we have:

$$(5.5.13)\qquad D_{k,r}-d_{(k;r)} \geq \{(180^a+120^a)30^b+(120^a+80^a)20^b + +(2\cdot 60^a+2\cdot 40^a-80^a-72^a-48^a-(60^a+40^a)\cdot \cdot(\tfrac{a+5}{a-1}))\cdot 60^b\}/(3^a+2^a)\cdot H\cdot 60^b .$$

By Lemma (5.4.21), $180^a 30^b - 80^a 60^b \geq 2\cdot 72^a 60^b$. Combining this with (5.5.13), we find:

$$D_{k,r}-d_{(k;r)} \geq \{72^a 60^b+((\tfrac{a-7}{a-1})\cdot 60^a-48^a)60^b\}/(3^a+2^a)\cdot H\cdot 60^b .$$

Since $a > 20$, we have

$$D_{k,r}-d_{(k;r)} \geq \frac{72^a\cdot 60^b}{3^a\cdot 120^a\cdot 60^b} = \frac{1}{5^a} = C .$$

Again, $\hat{f}(6) < C$ and $f(6) < C$, so $n_o < 6^a$ and $\hat{n}_o < 6^a$.

<u>Case VI</u>

$$2(\frac{\log 3}{\log 2} - \frac{1}{2})r-2(\frac{9}{10})^r/\log 2 < k \leq (\frac{\log 5}{\log 2})r - (\frac{5}{6})^r/\log 2 .$$

By Lemma (5.3.25), $d_{(k;r)} < 1 - \dfrac{1}{2^r} - \dfrac{4^r}{3^r(4^r+2^r)}$.

We must re-estimate $D_{k,r}$:

$$D_{k,r} > \frac{1 + \dfrac{1}{2^{a+b}} + \dfrac{1}{3^{a+b}}}{1 + \dfrac{1}{2^a} + \dfrac{1}{3^a} + \dfrac{1}{4^a} + \dfrac{1}{5^a} + \dfrac{1}{6^a} + (\dfrac{a+6}{a-1})\cdot \dfrac{1}{7^a}}$$

$$(5.5.14) \qquad = (420^{a+b} + 210^{a+b} + 140^{a+b})/420^b \cdot J ,$$

where

$$J = 420^a+210^a+140^a+105^a+84^a+70^a+(\frac{a+6}{a-1})60^a .$$

From Lemma (5.3.25) and (5.5.14), we find:

$$(5.5.15) \quad D_{k,r}-d_{(k;r)} \geq \{(5040^a+2520^a)3^b+(3360^a+1680^a)\cdot 2^b + - (2016^a+1440^a(\frac{a+6}{a-1}))6^b\}/2^a(12^a+6^a)\cdot 6^b \cdot J .$$

By Lemma (5.4.25), $5040^a 3^b - 2016^a 6^b \geq 1680^a 6^b$. Combining this with (5.5.15), we find:

$$D_{k,r}-d_{(k;r)} \geq \{1680^a-(\frac{a+6}{a-1})(1440^a)\}/2^a\cdot(12^a+6^a)\cdot J$$
$$\geq .94/6^a = C ,$$

since $a > 20$. Again, $\hat{f}(8) < C$ and $f(8) < C$, so $n_o < 8^a$ and $\hat{n}_o < 8^a$.

Case VII

$$(\frac{\log 5}{\log 2})r-(\frac{5}{6})^r/\log 2 < k \leq (\frac{\log 5}{\log 2})r + \frac{1}{2}(\frac{5}{6})^r/\log 2 .$$

By Lemma (5.3.28), $d_{(k;r)} < 1 - \dfrac{1}{2^r} - \dfrac{5^r}{3^r(5^r+2\cdot 2^r)}$.

Combining this with (5.5.14), we find:

$$(5.5.16) \quad D_{k,r}-d_{(k;r)} \geq \{(6300^a+2\cdot 2520^a)3^b+(4200^a+2\cdot 1680^a)\cdot 2^b + (2100^a-2\cdot 1680^a-2520^a-2\cdot 1008^a + (-1800^a-2\cdot 720^a)\cdot(\frac{a+6}{a-1}))\cdot 6^b\}/2^a(15^a+2\cdot 6^a)\cdot 6^b\cdot J .$$

By Lemma (5.4.29), $6300^a3^b - 2520^a6^b \geq -\frac{1}{2} \cdot 2100^a6^b$.

Combining this with (5.5.16), we find, since $a > 20$,

$$D_{k,r} - d(k;r) > .4 \cdot \frac{2100^a \cdot 6^b}{15^a \cdot 6^b \cdot 840^a}$$

$$= \frac{.4}{6^a} = C \ .$$

Thus $\hat{f}(8) < C$ and $f(8) < C$, so $n_o < 8^a$ and $\hat{n}_o < 8^a$.

Case VIII

$$\left(\frac{\log 5}{\log 2}\right) r + \frac{1}{2} \left(\frac{5}{6}\right)^r / \log 2 < k \leq \left(\frac{\log 6}{\log 2}\right) r \quad .$$

By Lemma (5.3.31), $d_{(k;r)} < 1 - \frac{1}{2^r} - \frac{5^r + 2 \cdot 3^r - \left(\frac{6}{5}\right)^r}{3^r(5^r+3^r+2^r)}$.

Combining this with (5.5.14), we find:

$$\begin{aligned} D_{k,r} - d(k;r) &> (2100^a + 1575^a - 2 \cdot 1008^a - 1680^a - 1512^a - 336^a - 252^a - 168^a + \\ &\quad -(201.6)^a - (144^a + 1800^a + 1080^a + 720^a)\left(\frac{a+6}{a-1}\right))/J \cdot 6^a(5^a + 3^a + 2^a) \\ &> .8/6^a = C \ , \end{aligned}$$

since $a > 20$. Thus $\hat{f}(8) < C$ and $f(8) < C$, and so $n_o < 8^a$ and $\hat{n}_o < 8^a$.

Case IX

$$k \geq \left(\frac{\log 6}{\log 2}\right) r \quad .$$

In this case, $2^k > 6^r$. We know (Orr [12]) that $d_r = \frac{q_r(n_r)}{n_r}$

for $5^r \leq n_r < 6^r$. But since $2^k > 6^r$, $q_{(k;r)}(n_r) = q_{k,r}(n_r) = q_r(n_r)$. Thus $d_{k,r} \leq d_r$ and $d_{(k;r)} \leq d_r$. But $Q_r \subset Q_{k,r} \subset Q_{(k;r)}$ implies $d_r \leq d_{k,r} \leq d_{(k,r)}$. Thus $d_{k,r} = d_{(k;r)} = d_r = (q_r(n_r))/n_r$.

Q.E.D.

Remark: Although Case VIII is valid for $k \geq (\frac{\log 6}{\log 2})r$, the above argument gives a better result.

We shall use an argument similar to Case IX above to prove the following useful result.

Theorem (5.5.17). For $2 \leq r < k \leq \infty$, if $d_{k,r} = q_{k,r}(n_o)/n_o$ and $2^{k+r} > n_o$, then $d_{(k;r)} = d_{k,r} = q_{k,r}(n_o)/n_o = q_{(k;r)}(n_o)/n_o$.

Proof: If $2^{k+r} > n_o$, then $q_{(k;r)}(n_o) = q_{k,r}(n_o)$. But $Q_{k,r} \subset Q_{(k;r)}$. Thus for $n > n_o$, $q_{(k;r)}(n)/n \geq q_{k,r}(n)/n \geq d_{k,r} =$ $= q_{k,r}(n_o)/n_o$. But if $n \leq n_o$, $q_{(k;r)}(n)/n = q_{k,r}(n)/n \geq d_{k,r} =$ $= q_{k,r}(n_o)/n_o$. Thus $d_{(k;r)} = q_{k,r}(n_o)/n_o = q_{(k;r)}(n_o)/n_o$.

Q.E.D.

We shall now categorize more precisely the intervals in which $d_{k,r}$ and $d_{(k;r)}$ must be achieved, and apply the above result to conclude that often $d_{k,r} = d_{(k;r)}$.

Theorem (5.5.18). Let $20 < r < k \leq \infty$. We have:

i) If $2^k < \dfrac{15^r-9^r}{3^r+5^r}$, then $n_{k,r} = n_{(k;r)} = 2^r$.

ii) If $\dfrac{15^r-9^r}{3^r+5^r} < 2^k < \dfrac{15^r}{3^r+5^r}$, then $n_{(k;r)} = 2^r$ and

$n_{k,r} \in [2^r,6^r)$.

iii) If $\dfrac{15^r}{3^r+5^r} < 2^k < 3^r$, then $n_{(k;r)} \in [2^r,6^r)$ and

$n_{k,r} \in [2^r,6^r)$.

iv) If $3^r < 2^k < 3^r + 2^r$, then $n_{k,r} = n_{(k;r)} \in [3^r,6^r)$.

v) If $3^r + 2^r < 2^k < 2\cdot 3^r$, then $n_{k,r} = n_{(k;r)} \in [3^r,2^k)$.

vi) If $2\cdot 3^r < 2^k < \frac{1}{2}\cdot 5^r$, then $n_{k,r} = n_{(k;r)} \in [\frac{1}{2}\cdot 2^k,2^k)$.

vii) If $\frac{1}{2}\cdot 5^r < 2^k < 5^r$, then $n_{k,r} = n_{(k;r)} \in ([\frac{1}{2}\cdot 2^k,2^k) \cup$

$\cup [5^r,2\cdot 2^k) \cup [2\cdot 5^r,3\cdot 2^k) \cup \ldots \cup [m\cdot 5^r,(m+1)\cdot 2^k)) \cap (1,6^r)$

where m is the largest integer such that $m\cdot 5^r < (m+1)\cdot 2^k$.

viii) If $5^r < 2^k < 5^r + 3^r + 2^r$, then $n_{k,r} = n_{(k;r)} \in [5^r,7^r)$.

ix) If $5^r + 3^r + 2^r < 2^k < 2\cdot 5^r$, then $n_{k,r} = n_{(k;r)} \in [5^r,2^k)$.

x) If $2\cdot 5^r < 2^k < 6^r$, then $n_{k,r} = n_{(k;r)} \in [\frac{1}{2}\cdot 2^k,2^k)$.

xi) If $6^r < 2^k$, then $n_{k,r} = n_{(k;r)} = n_r \in [\frac{1}{2}\cdot 6^r,6^r)$.

<u>Remark</u>: Since k and r are integers, it can easily be shown that equality can never occur in any of the conditions above. Whenever we assert $n_{k,r} = n_{(k;r)}$ it follows that $d_{k,r} = d_{(k;r)}$.

<u>Proof</u>. By Theorem (5.5.1) we know $n_{k,r} \in [2^r, 8^r)$ and $n_{(k;r)} \in [2^r, 8^r)$. By Lemma (4.2.3), we have

$$q_{(k;r)}(s) = \sum_{\substack{(a,b)=1 \\ a \in Q_2}} \mu(b) \left[\frac{s}{a^k b^r} \right] .$$

But if n is a possible value of $n_{k,r}$ or $n_{(k;r)}$, we may assume $n < 8^r$. Thus

$$(5.5.19) \qquad q_{(k;r)}(n) = n - \left[\frac{n}{2^r} \right] - \left[\frac{n}{3^r} \right] - \left[\frac{n}{5^r} \right] - \left[\frac{n}{7^r} \right] + \left[\frac{n}{6^r} \right] + + \left[\frac{n}{2^k} \right] + \left[\frac{n}{3^k} \right] + \left[\frac{n}{5^k} \right] + \left[\frac{n}{6^k} \right] + + \left[\frac{n}{7^k} \right] - \left[\frac{n}{2^k \cdot 3^r} \right] - \left[\frac{n}{2^r \cdot 3^k} \right] .$$

As was stated by Feng [9]. Lemma (2.1.1), with $n < 8^r$

$$q_{k,r}(n) = q_r(n) + q_r\left(\frac{n}{2^k} \right) + \dots + q_r\left(\frac{n}{7^k} \right) .$$

But, by Lemma (4.2.3), $(k = \infty)$,

$$q_r(n) = \sum_{b=1}^{\infty} \mu(b) \left[\frac{n}{b^r} \right] .$$

Thus, since $n < 8^r$:

$$q_{k,r}(n) = n - [\tfrac{n}{2^r}] - [\tfrac{n}{3^r}] - [\tfrac{n}{5^r}] - [\tfrac{n}{7^r}] + \tag{5.5.20}$$

$$+ [\tfrac{n}{6^r}] + [\tfrac{n}{2^k}] - [\tfrac{n}{2^k \cdot 2^r}] - [\tfrac{n}{2^k \cdot 3^r}] +$$

$$+ [\tfrac{n}{3^k}] - [\tfrac{n}{3^k \cdot 2^r}] + [\tfrac{n}{4^k}] + [\tfrac{n}{5^k}] +$$

$$+ [\tfrac{n}{6^k}] + [\tfrac{n}{7^k}] \ .$$

We shall now prove the theorem case by case.

<u>Cases i), ii) and iii)</u>

We start with $n_{(k;r)}$. First, let us assume $2^k < \frac{1}{2} \cdot 3^r$. Then, for $n \geq 2^k$, it can easily be shown (since $r > 20$) that

$$[\tfrac{n}{2^k}] > [\tfrac{n}{3^r}] + [\tfrac{n}{5^r}] + [\tfrac{n}{7^r}] + [\tfrac{n}{2^k \cdot 3^r}] + [\tfrac{n}{2^r \cdot 3^k}] \ .$$

Thus, for $n \geq 2^k$, by (5.5.19) we have

$$q_{(k;r)}(n)/n > (n - [\tfrac{n}{2^r}])/n \geq 1 - 2^{-r} \ .$$

For $n < 2^k$,

$$q_{(k;r)}(n)/n = (n - [\tfrac{n}{2^r}])/n \geq 1 - 2^{-r} \ .$$

But $q_{(k;r)}(2^r)/2^r = 1 - 2^{-r}$. Thus $n_{(k;r)} = 2^r$.

Now let $2^k > \frac{1}{2} \cdot 3^r$. Then, since $n < 8^r$,

$$(5.5.21) \qquad [\frac{n}{5^k}] = [\frac{n}{6^k}] = [\frac{n}{7^k}] = [\frac{n}{2^k \cdot 3^r}] = [\frac{n}{2^r \cdot 3^k}] = 0 \quad .$$

We also have

$$[\frac{n}{6^r}] \geq [\frac{n}{7^r}] \quad .$$

If we also have $2^k < \frac{15^r}{3^r+5^r}$, a simple calculation shows that $[\frac{n}{2^k}] \geq [\frac{n}{3^r}] + [\frac{n}{5^r}]$. Thus

$$q_{(k;r)}(n)/n \geq (n - [\frac{n}{2^r}])/n \geq 1-2^{-r}$$

and again $n_{(k;r)} = 2^r$.

Now let $\frac{15^r}{3^r+5^r} < 2^k < 3^r$. Then r satisfies Case I or Case II from Theorem (5.5.1). If k satisfies Case II, we know from the proof of Theorem (5.5.1) that $n_{(k;r)} \in [2^r, 6^r)$. Let us assume k satisfies Case I. Then, from the proof of Case I, it can be shown that if $a > 100$, $\hat{f}(6) < C$, where $C = (5.705)^{-a} + - \frac{6}{a-1} \cdot 6^{-a} - (7.5)^{-a}$, and thus for $r > 100$, $n_{(k;r)} \in [2^r, 6^r)$. But since k and r are integers, there is no k such that for $20 < r \leq 100$, $\frac{15^r}{3^r+5^r} < 2^k < 3^r$. Thus Cases i) - iii) hold for $n_{(k;r)}$.

We now consider $n_{k,r}$. Let us first assume $2^k < \frac{1}{2} \cdot 3^r$. Then for $n \geq 2^k$, it can easily be shown that

$$[\frac{n}{2^k}] > [\frac{n}{3^r}] + [\frac{n}{5^r}] + [\frac{n}{7^r}] + [\frac{n}{2^k \cdot 3^r}] +$$

$$+ [\frac{n}{2^r \cdot 3^k}] + [\frac{n}{2^{k+r}}] \quad .$$

Then, arguing as for $n_{(k;r)}$, we have $n_{k,r} = 2^r$.

Now let $n > \frac{1}{2} \cdot 3^r$. Then (5.5.21) holds and $[\frac{n}{4^k}] = 0$. Furthermore, $[\frac{n}{6^r}] \geq [\frac{n}{7^r}]$. If $2^k < \frac{15^r - 9^r}{3^r + 5^r}$, then a simple calculation shows

$$[\frac{n}{2^k}] \geq [\frac{n}{3^r}] + [\frac{n}{5^r}] + [\frac{n}{2^{k+r}}] \quad .$$

Thus $q_{k,r}(n)/n \geq (n - [\frac{n}{2^r}])/n \geq 1 - 2^{-r}$, and we have $n_{k,r} = 2^r$. If $\frac{15^r - 9^r}{3^r + 5^r} < 2^k < 3^r$, we can argue as for $n_{(k;r)}$ that for $r > 100$, $n_{k,r} \in [2^r, 6^r)$. Finally, if $20 < r \leq 100$, there is no k such that $\frac{15^r - 9^r}{3^r + 5^r} < 2^k < 3^r$. This finishes Cases i) - iii).

<u>Case iv)</u>

That $n_{k,r} < 6^r$ and $n_{(k;r)} < 6^r$ follows from the proof of Theorem (5.5.1), Cases II, III and IV. Since $2^{k+r} > 3^r \cdot 2^r = 6^r$ it follows from Theorem (5.5.17) that $n_{(k;r)} = n_{k,r}$. We know that

$$q_{k,r}(3^r)/3^r = q_{(k;r)}(3^r)/3^r < 1 - 2^{-r}$$

while if $n < 3^r$,

$$q_{k,r}(n)/n = q_{(k;r)}(n)/n \geq 1 - 2^{-r} \ .$$

Thus $n_{k,r} \geq 3^r$ and $n_{(k;r)} \geq 3^r$.

Case v)

It follows from the proof of Theorem (5.5.1), Cases IV and V, that $n_{k,r} < 6^r$ and $n_{(k;r)} < 6^r$. Since $2^{k+r} > 3^r \cdot 2^r = 6^r$, we have $2^{k+r} > n_{k,r}$ and by Theorem (5.5.17) $n_{k,r} = n_{(k;r)}$. We shall therefore consider $n_{(k;r)}$ only.

As in Case iv), $n_{(k;r)} \geq 3^r$. Since any n which could be $n_{(k;r)}$ must satisfy $n < 6^r$ and since $2^k > 3^r$, (5.5.19) reduces to:

$$q_{(k;r)}(n) = n - [\frac{n}{2^r}] - [\frac{n}{3^r}] - [\frac{n}{5^r}] + [\frac{n}{2^k}] + [\frac{n}{3^k}] \ .$$

Let $\tilde{n} = 2^r([\frac{3^r}{2^r}] + 1) < 2^r + 3^r < 2^k$. Then

$$\begin{aligned} q_{(k;r)}(\tilde{n})/\tilde{n} &= 1 - 2^{-r} - 1/\tilde{n} \\ &< 1 - 2^{-r} - 1/(3^r+2^r) \ . \end{aligned}$$

But for $2^k < 2 \cdot 3^r$, if $n \geq 2^k$, then

$$([\frac{n}{2^k}] - [\frac{n}{5^r}]) > \frac{1}{2} \cdot (\frac{n}{2^k}) > \frac{1}{4} \cdot \frac{n}{3^r} \ .$$

Thus if $n \geq 2^k$,

$$q_{(k;r)}(n)/n \geq (n - [\frac{n}{2^r}] - [\frac{n}{3^r}] - [\frac{n}{5^r}] + [\frac{n}{2^k}])/n$$

$$> 1 - 2^{-r} - 3^{-r} + 1/4\cdot 3^r$$

$$> 1 - 2^{-r} - 1/(3^r+2^r) \quad .$$

Thus $n_{(k;r)} < 2^k$ and Case v) follows.

<u>Case vi)</u>

It follows from Theorem (5.5.1) that $n_{k,r} < 8^r$. Since $2^k > 2\cdot 3^r$, we have, from (5.5.20)

$$q_{k,r}(n) = n - [\frac{n}{2^r}] - [\frac{n}{3^r}] - [\frac{n}{5^r}] - [\frac{n}{7^r}] + [\frac{n}{6^r}]$$

$$+ [\frac{n}{2^k}] - [\frac{n}{2^k\cdot 2^r}] - [\frac{n}{2^k\cdot 3^r}] + [\frac{n}{3^k}] - [\frac{n}{3^k\cdot 2^r}].$$

But $[\frac{n}{6^r}] \geq [\frac{n}{2^k\cdot 3^r}] + [\frac{n}{7^r}] + [\frac{n}{3^k\cdot 2^r}]$, since $r > 20$. Also, for $2^k < \frac{1}{2}\cdot 5^r$, if $n \geq 2^k$,

$$([\frac{n}{2^k}] - [\frac{n}{3^k}] - [\frac{n}{5^r}] - [\frac{n}{2^k\cdot 2^r}]) > \frac{1}{3}\cdot(\frac{n}{2^k}) \; .$$

Thus if $n \geq 2^k$,

$$q_{k,r}(n)/n \geq (n - [\frac{n}{2^r}] - [\frac{n}{3^r}] + \frac{1}{3}(\frac{n}{2^k}))/n$$

$$\geq 1 - 2^{-r} - 3^{-r} + \frac{1}{3}\cdot 2^{-k} \quad .$$

If $2^k < 2\cdot 4^r$ we define $\tilde{n}$ as for Case v) and observe:

$$q_{k,r}(\tilde{n})/\tilde{n} = 1 - 2^{-r} - 1/\tilde{n} < 1 - 2^{-r} - 1/(3^r+2^r)$$

and if $n \geq 2^k$,

$$q_{k,r}(n)/n \geq 1 - 2^{-r} - 3^{-r} + \frac{1}{6} 4^{-k} > q_{k,r}(\tilde{n})/\tilde{n} \quad .$$

Thus if $2^k < 2 \cdot 4^r$, $n_{k,r} < 2^k$.

Now if $2^k \geq 2 \cdot 4^r$, we define $\tilde{n}$ as for Lemma (5.3.25) and observe:

$$q_{k,r}(\tilde{n})/\tilde{n} < 1 - 2^{-r} - 4^r/(3^r(4^r+2^r))$$

and if $n \geq 2^k$, since in Case vi) $2^k < \frac{1}{2} \cdot 5^k$,

$$q_{k,r}(n)/n \geq 1 - 2^{-r} - 3^{-r} + \frac{1}{3} 5^{-k} > q_{k,r}(\tilde{n})/\tilde{n} \quad .$$

Thus, always in Case vi), $n_{k,r} < 2^k < \frac{1}{2} \cdot 5^r$. But $2^k \cdot 2^r > 3^r \cdot 2^r = 6^r$ and so $2^{k+r} > n_{k,r}$. Applying Theorem (5.5.17), we find $n_{k,r} = n_{(k;r)}$.

Arguing as in Diananda and Subbarao [2], we see that if $n_{(k;r)} < \frac{1}{2} \cdot 2^{-k}$,

$$q_{(k;r)}(2 \cdot n_{(k;r)})/2 \cdot n_{(k;r)} = q_{(k;r)}(n_{(k;r)})/n_{(k;r)}$$

and thus we may assume $n_{(k;r)} \geq \frac{1}{2} \cdot 2^{-k}$. This completes Case vi).

Case vii)

By Theorem (5.5.1), we may assume $n_{k,r} < 8^r$. Since $2^k > \frac{5^r}{2}$, it follows from (5.5.20) that

$$q_{k,r}(n) = n - [\frac{n}{2^r}] - [\frac{n}{3^r}] - [\frac{n}{5^r}] - [\frac{n}{7^r}] + [\frac{n}{6^r}] + [\frac{n}{2^k}] .$$

Let $\tilde{n} = \left[\dfrac{\frac{1}{4} \cdot 5^r}{3^r} \right] \cdot 3^r + 3^r$ and $\tilde{\tilde{n}} = [\dfrac{\tilde{n}}{2^r}] \cdot 2^r + 2^r$. Then

$$q_{(k;r)}(\tilde{\tilde{n}})/\tilde{\tilde{n}} = 1 - 2^{-r} - [\frac{\tilde{\tilde{n}}}{3^r}]/\tilde{\tilde{n}}$$

$$< 1 - 2^{-r} - 5^r/3^r(5^r+4\cdot 2^r) \quad .$$

Now $[\dfrac{n}{6^r}] \geq [\dfrac{n}{7^r}]$ and $[\dfrac{n}{2^k}] \geq [\dfrac{n}{5^r}]$. Furthermore, if $n > 6^r$, $[\dfrac{n}{6^r}] - [\dfrac{n}{7^r}] \geq \dfrac{n}{3\cdot 6^r}$, while if

$$n \notin ([1,2^k) \cup [5^r,2\cdot 2^k) \cup [2\cdot 5^r,3\cdot 2^k) \cup \ldots \cup [m\cdot 5^r,(m+1)\cdot 2^k))$$

$$[\frac{n}{2^k}] - [\frac{n}{5^r}] \geq 1 \; .$$

Thus, if

$$n \notin (([1,2^k) \cup [5^r,2\cdot 2^k) \cup \ldots \cup [m\cdot 5^r,(m+1)\cdot 2^k)) \cap [1,6^r)) \; ,$$

we have

$$q_{k,r}(n)/n \geq 1 - \frac{1}{2^r} - \frac{1}{3^r} + \frac{1}{3\cdot 6^r} > 1-2^{-r}-5^r/(3^r(5^r+4\cdot 2^r)) \; .$$

Thus

$$n_{k,r} \in (([1,2^k) \cup [5^r,2\cdot 2^k) \cup \ldots \cup [m\cdot 5^r,(m+1)\cdot 2^k)) \cap [1,6^r)) \; .$$

Since $2^k \cdot 2^r > 6^r$, we may apply Theorem (5.5.17) to conclude $n_{(k;r)} = n_{k,r}$. Arguing as in Case vi), we may assume $n_{(k;r)} = n_{k,r} \geq \frac{1}{2} \cdot 2^k$. This completes Case vii).

Case viii)

To prove this case, we tighten the argument for Theorem (5.5.1), Case VII, which covers all k from Case viii). We observe that for $a > 35$, $f(7) < C = \frac{.4}{6^a}$. Thus for $r > 35$, $n_{k,r} < 7^r$. But since k and r are integers, there is no k for $20 < r \leq 35$ such that $5^r < 2^k < 5^r + 3^r + 2^r$. Since $2^k + 2^r > 7^r$, we can conclude $n_{(k;r)} = n_{k,r}$. This completes this case.

Cases ix) and x)

By Theorem (5.5.1), $n_{k,r} < 8^r < 2^{k+r}$, since $2^k > 5^r$. Thus by Theorem (5.5.17), $n_{k,r} = n_{(k;r)}$. We shall consider $n_{(k;r)}$. Defining s_2 as in Lemma (5.3.31), we find

$$q_{(k;r)}(s_2)/s_2 < 1 - 2^{-r} - \frac{1}{5^r+3^r+2^r} - \frac{5^r}{3^r(5^r+2^r)} .$$

Since $2^k > 5^r$, from (5.5.19) we obtain

$$q_{(k;r)}(n) = n - [\frac{n}{2^r}] - [\frac{n}{3^r}] - [\frac{n}{5^r}] - [\frac{n}{7^r}] + [\frac{n}{6^r}] + [\frac{n}{2^k}] .$$

Again, $[\frac{n}{6^r}] \geq [\frac{n}{7^r}]$. If $n < 5^r$, we have

$$\begin{aligned} q_{(k;r)}(n)/n &= 1 - [\frac{n}{2^r}]/n - [\frac{n}{3^r}]/n \\ &> 1 - 2^{-r} - 3^{-r} \\ &> q_{(k;r)}(s_2)/s_2 . \end{aligned}$$

If $n \geq 2^k$, $[\frac{n}{2^k}] \geq \frac{1}{2} \cdot (\frac{n}{2^k})$, and

$$q_{(k;r)}(n)/n \geq 1 - [\frac{n}{2^k}]/n - [\frac{n}{3^k}]/n - [\frac{n}{5^k}]/n + [\frac{n}{2^k}]/n$$

$$> 1 - 2^{-r} - 3^{-r} - 5^{-r} + \frac{1}{2 \cdot 2^k}$$

$$> 1 - 2^{-r} - 3^{-r} - 5^{-r} + \frac{1}{2 \cdot 6^r}$$

$$> q_{(k;r)}(s_2)/s_2 \quad .$$

Thus $n_{(k;r)} \in [5^r, 2^k)$. If $2^k > 2 \cdot 5^r$, then we may argue as in Case vi) to prove that we may assume $n_{(k;r)} \in [\frac{1}{2} \cdot 2^k, 2^k)$. This finishes Cases ix) and x).

Case xi)

This was proved in Theorem (5.5.1), Case IX and is included here for completeness.

Q.E.D.

Using Lemmas (5.2.1) and (5.2.2) together with computer searches, we evaluated $n_{k,r}$ and $q_{k,r}(n_{k,r})$ for all $2 \leq r \leq 20$ and all k, $k \geq r$. We also evaluated $n_{(k;r)}$ and $q_{(k;r)}(n_{(k;r)})$ on the same range, except for $n_{(2;3)}$. We found that (except for $k = 3$ and $r = 2$) $n_{(k;r)} = n_{k,r}$ and $d_{k,r} = d_{(k;r)}$. The values obtained are listed in Table 4. (The values of $n_{k,r}$ and $q_{k,r}(n_{k,r})$ for $r = 2$, $k \geq 4$ are due to Diananda [3].) A computer search

for $1 \leq n \leq 2^{31} - 1 = 2\ 147\ 483\ 647$ failed to discover an n such that $q_{(3;2)}(n)\ n^{-1} < D_{(3;2)}$. Thus the question of whether or not $d_{(3;2)} < D_{(3;2)}$ remains open.

<u>Remark</u>: From the results in Table 4, we can conclude that Theorems (5.5.1) and (5.5.18) are both valid for all $5 \leq r < k \leq \infty$.

CHAPTER VI

SOME OPEN PROBLEMS

The investigations carried out in this disssertation relate mainly to two specific areas of research: 1) The Schnirelmann density of the r-free integers, of the (k,r)-free integers, and of the (k,r) integers; 2) Certain asymptotic properties of the (k,r)-free integers and the unification of the study of the r-free and semi r-free integers. In both areas, there are many open questions.

We have already commented on two open questions in the Schnirelmann density problem, namely the extension of the calculations of d_k beyond $k = 75$ and finding a proof (or counterproof) for the conjectures from (§4), Chapter II. In the light of Conjecture (2.4.5), (which we believe is very likely to be correct), we are severely restricted in any attempt to extend the calculation of d_k since we must find a minimum for E_k over $(\frac{6}{5})^k$ points. A proof of the conjectures from (§4), Chapter II would be important since this would characterize the behavior of d_k and would effectively solve the density problem.

A number of related problems arise. The first concerns the uniqueness of n_k. There are four known values of k for which n_k is not unique, namely $k = 5, 38, 55, 56$ (from among $k \in \{5, 6, \ldots, 63\}$). 38 is known to have at least a three fold repetition of n_{38}. This leads us to ask the following questions:

a) Are there infinitely many k such that n_k is unique?

b) Are there infinitely many k such that n_k is not unique?

c) Although we know that the number of repetitions of n_k is finite for k fixed, (since $n_k < 6^k$), is the $\sup_k \{j_k\} < \infty$, where j_k is the number of repetitions of n_k?

d) What is the asymptotic density of the set of k such that n_k is unique?

Although we have not been able to provide answers to any of the above questions, not even for the mild question a) (for which we feel undoubtedly the answer is yes), we will conjecture:

Conjecture (6.1): The set of k such that n_k is unique has positive asymptotic density.

Another problem arises as follows: Consider an arbitrary set (finite or infinite) of primes $S = \{p_1, p_2, p_3, \ldots\}$ and another set of positive integers $T_r = \{n : p^r \nmid n \text{ for any } p \in S\}$, where $r \geq 1$ is fixed. We ask: Does T_r have the same asymptotic and Schnirelmann densities.

If S is finite, and if S has more than 1 element, then clearly T_r has different asymptotic and Schnirelmann densities. Indeed, if $S = \{p_1, p_2, \ldots, p_k\}$ then the asymptotic density D of T_r will be $D = \prod_{i=1}^{k} (1 - \frac{1}{p_i})$. If $q(n)$ is the number of elements of T_r which are less than or equal n and $N = \prod_{i=1}^{k} p_i$, then $q(N)/N = D$. If p_j is the smallest element of S, $q(N-p_j) = q(N)-(p_j-1)$. But then $q(N-p_j)/(N-p_j) = (ND-(p_j-1))/(N-p_j) < D$. Thus

the Schnirelmann density is less than the asymptotic density.

The case for S infinite is more complicated. We note that if $S = \{p : p \geq p_0\}$, a straight foreward modification of Stark's [16] proof that $d_r \neq D_r$ proves T_r has different asymptotic and Schnirelmann densities. However, not all infinite S give rise to T_r with different asymptotic and Schnirelmann densities. Carl Pomerance proved there exist $p_1 < p_2 < \cdots$ such that $\sum_{i=1}^{\infty} p_i^{-1} < \infty$ but $T = \{n : p_i \nmid n \text{ for all } i\}$ has equal asymptotic and Schnirelmann densities.

We can also consider the problem of finding the density of the k-free (and (k,r)-free) numbers where the density is defined in an alternate manner. One such alternate definition, due to Stanlley [15] is:

$$\alpha_k^* = \underset{n \in Q_k}{\text{g.}\ell\text{.b.}} \frac{q_k(n)}{n} .$$

There are other densities described in the literature which could also be considered.

More open problems arise from generalizing the (k,r)-free integers even further. One possible generalization, suggested by Professor L. Carlitz, is as follows: Given $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ such that $1 \leq \beta_i < \alpha_i \leq \infty$, and the sequence of primes $2, 3, 5, \ldots = p_1, p_2, p_3, \ldots,$ and any integer $n = p_{\gamma_1}^{\delta_1} \cdot p_{\gamma_2}^{\delta_2} \cdot \ldots \cdot p_{\gamma_i}^{\delta_i}$ then define n to be (α,β)-free if $\delta_j < \beta_{\gamma_j}$ or $\delta_j \geq \alpha_{\gamma_j}$ for $1 \leq j \leq i$. Given that all but a finite number of the $\beta_i \geq r$, then it is fairly straight foreward to show

that if we define $T_{(\alpha;\beta)}(n)$ to be the number of (α,β)-free integers which are $\leq n$, then

$$T_{(\alpha;\beta)}(n) = n \cdot \prod_{k=1}^{\infty} \left(1 - \frac{1}{p_i^{\alpha_i}} + \frac{1}{p_i^{\beta_i}} \right) + O(n^{1/r}) \qquad \text{as} \quad n \to \infty$$

where the constant implied by the O-estimate depends on α and β. We ask if other properties of the (k,r)-free integers can be similarly generalized?

Another generalization of the (k,r)-free integers arises from a paper by Erdös, Babu and Ramachandra [6]. Their result was as follows: Let $b_1 < b_2 < \ldots$ such that $(b_i, b_j) = 1$ for $i \neq j$ and $\sum b_i^{-1} < \infty$ $(s = \{b_i\})$. Let $T = \{n : n$ is not divisible by any element of $s\}$. Then if $N(n) =$ the number of representations of n as the sum of a prime and an element of T, then

$$N(n) = n\,(\log n)^{-1} \prod_{(b_j,b)=1} (1 - (\phi(b_j))^{-1}) + \\ + O(n(\log n)^{-2}) \,, \qquad \text{as} \quad n \to \infty \,.$$

We propose the following generalization to their problem: Given s as above, let us assume that we are given $\hat{s} = \{\hat{b}_1, \hat{b}_2, \ldots\}$ such that $b_i | \hat{b}_i$, $\gamma(b_i) = \gamma(\hat{b}_i)$ and $b_i < \hat{b}_i$. (This implies that $(\hat{b}_i, \hat{b}_j) = 1$ whenever $i \neq j$). Let $T(s,\hat{s})$ be the set given by $T(s,\hat{s}) = \{n :$ for any fixed i, either $b_i \nmid n$ or $\hat{b}_i | n\}$. Then we expect the number of representations of n as the sum of a prime and an element of $T(s,\hat{s})$ will be given by

$$N(m) \sim m(\log m)^{-1} \prod_{(b_k,m)=1} (1 - (\phi(b_j))^{-1} + (\phi(\hat{b}_j)^{-1})) \quad \text{as} \quad m \to \infty \ .$$

We pose as an open question whether this and other results obtained for the (k,r)-free integers can be generalized to integers belonging to $T(s,\hat{s})$.

The following papers, while not related directly to the results in this dissertation, do consider some additive properties of k-free and generalized k-free integers which may yield analogous results for the (k,r)-free integers:

1. L. Carlitz, A problem in additive arithmetic, Quarterly J. Math. Oxford Series, 2 (1931), 97-106.
2. L. Carlitz, A problem in additive arithmetic II, Quarterly J. Math. Oxford Series, 3 (1932), 273-290.
3. E. Cohen, Some sets of integers related to the k-free integers, Acta Sci. Math. Szeged, 22 (1961), 223-233.
4. E. Cohen, The average order of certain types of arithmetic functions: generalized k-free numbers and totient points, Monatsh Math. 64 (1960), 251-262.
5. E. Cohen, Remark on a set of integers, Acta Sci. Math. Szeged, 25 (1964), 179-181.
6. E. Cohen and R.L. Robinson, On the distribution of k-free integers in residue classes, Acta Arith. 8 (1963), 283-293.
7. G. Reiger, Einge verteilungsfragen mit K-leeren zahlen, r-zahlen und primzahlen, J. Riene Angew Math. 262/263 (1973), 189-193.

8. M.V. Subbarao and Y.K. Feng, Representation and distribution problems involving (k-r)-integers, Amer. Math. Soc. Not. 17 (1970), 277-278.

9. M.V. Subbarao and D. Suryanarayana, Some theorems in additive number theory, Annales Univ. Ser. Bud. 15 (1972), 5-16.

10. M.V. Subbarao and D. Suryanarayana, The divisor problem for (k,r)-integers, J. Austrailian Math. Soc. 15 (1973), 430-440.

APPENDIX A

TABLES AND GRAPHS

Table 1

TABLE OF VALUES OF n_k

k	$q_k(n_k)$	n_k
2	106	176
3	314	378
4	2 320	2 512
5	6 110	6 336
6	30 825	31 360
7	234 331	236 288
8	1 169 758	1 174 528
9	7 798 488	7 814 151
10	48 785 015	48 833 536
11	292 856 489	293 001 216
12	1 709 225 206	1 709 645 824
13	12 206 236 915	12 207 734 784
14	67 139 207 400	67 143 319 552
15	366 201 607 242	366 212 808 704
16	2 593 955 782 238	2 593 995 423 744
17	15 258 697 717 317	15 258 814 251 008
18	83 923 093 402 988	83 923 413 762 048
19	476 836 615 418 082	476 837 525 323 776
20	3 337 857 168 384 426	3 337 860 352 573 440
21	11 920 926 635 354 486	11 920 932 320 837 632
22	100 135 807 616 763 580	100 135 831 494 197 248

k	$q_k(n_k)$	n_k
23	751 018 458 383 613 488	751 018 547 919 978 496
24	3 099 441 423 652 148 001	3 099 441 608 404 238 336
25	27 418 136 574 859 993 489	27 418 137 392 016 523 264
26	96 857 546 842 298 241 268	96 857 548 285 626 286 080
27	730 156 894 087 760 708 214	730 156 899 527 949 287 424
28	4 656 612 859 882 900 146 794	4 656 612 877 230 338 473 984
29	26 822 090 128 376 729 645 915	26 822 090 178 337 156 628 480
30	212 341 546 852 956 928 476 582	212 341 547 050 716 436 103 168
31	1 108 273 863 395 419 430 927 256	1 108 273 863 911 501 459 357 696
32	7 613 562 045 841 775 337 850 444	7 613 562 047 614 449 998 626 816
33	25 262 124 833 820 295 233 777 308	25 262 124 836 761 198 170 996 736
34	175 787 135 949 513 502 233 670 439	175 787 135 959 745 670 777 143 296
35	948 784 872 863 616 282 998 894 234	948 784 872 891 229 576 043 167 744
36	9 880 750 439 995 132 481 304 286 769	9 880 750 440 138 916 389 931 974 656

k	$q_k(n_k)$	n_k
37	49 622 030 928 687 145 579 945 860 190	49 622 030 929 048 193 483 916 967 936
38	190 993 887 372 205 292 034 200 123 988	190 993 887 372 900 123 890 134 548 480
39	1 711 669 028 733 996 013 101 990 741 984	1 711 669 028 737 109 521 350 009 028 608
40	12 878 444 977 093 042 082 277 489 144 174	12 878 444 977 104 754 960 810 526 113 792
41	53 887 561 080 022 183 204 170 287 473 304	53 887 561 080 046 688 431 294 635 835 392
42	374 711 817 130 362 649 038 419 878 188 058	374 711 817 130 447 848 644 937 399 664 640
43	2 810 338 628 478 039 366 316 870 491 079 793	2 810 338 628 478 358 864 837 030 497 484 800
44	9 129 053 069 045 285 647 147 120 161 235 982	9 129 053 069 045 804 573 743 796 707 131 392
45	52 096 993 385 929 460 407 350 299 177 991 715	52 096 993 385 930 941 092 976 146 004 312 064
46	581 081 849 304 604 721 761 373 050 523 576 363	581 081 849 304 612 979 431 176 722 400 149 504
47	3 616 662 525 018 885 461 322 898 969 131 549 402	3 616 662 525 018 911 159 255 883 470 776 303 616
48	22 353 674 467 012 678 623 674 139 681 398 618 292	22 353 674 467 012 758 039 879 470 330 199 146 496
49	106 954 445 300 288 890 870 463 750 358 352 526 465	106 954 445 300 289 080 859 724 610 735 207 088 128

k	$q_k(n_k)$	n_k
50	640 199 004 919 849 713 224 546 255 855 658 002 606	640 199 004 919 850 281 835 487 630 929 749 344 256
51	4 250 377 827 474 948 103 177 098 537 867 437 674 904	4 250 377 827 474 949 990 724 031 478 972 897 296 384
52	28 146 374 120 296 962 704 626 151 320 896 083 874 135	28 146 374 120 296 968 954 376 677 290 656 474 857 472
53	155 164 769 921 611 860 949 566 606 541 838 459 210 499	155 164 769 921 611 878 176 316 632 310 781 191 389 184
54	1 034 505 814 345 720 816 588 532 261 646 987 753 750 714	1 034 505 814 345 720 874 015 140 989 198 210 275 737 600
55	4 347 355 808 675 956 606 623 280 506 363 617 446 164 270	4 347 355 808 675 956 727 286 643 407 033 185 240 875 008
56	26 245 672 302 138 700 388 440 222 632 024 902 084 262 579	26 245 672 302 138 700 752 672 093 764 688 544 398 311 424
57	224 313 623 231 608 970 320 217 187 708 873 504 968 074 143	224 313 623 231 608 971 876 705 620 656 898 856 703 754 240
58	1 292 854 712 175 994 788 473 982 471 501 025 055 401 652 610	1 292 854 712 175 994 792 959 473 312 253 381 309 097 836 544
59	4 752 101 490 090 865 346 229 545 886 029 648 639 775 176 977	4 752 101 490 090 865 354 473 127 901 450 942 827 961 778 176

k	$q_k(n_k)$	n_k
60	45 369 957 790 697 412 708 967 510 350 004 955 325 769 420 605	45 369 957 790 697 412 748 319 675 792 875 042 264 378 769 408
61	226 849 788 953 487 063 643 217 965 358 091 885 107 791 131 942	226 849 788 953 487 063 741 598 378 964 375 211 321 893 847 040
62	1 441 186 579 798 081 623 691 977 022 964 454 018 964 834 267 015	1 441 186 579 798 081 624 004 484 547 123 038 279 659 977 441 280
63	9 918 931 995 200 885 863 918 169 838 800 856 053 807 764 762 609	9 918 931 995 200 885 864 993 582 600 602 800 146 637 658 259 456
64	46 898 249 173 032 979 794 912 462 909 389 414 781 510 858 116 634	46 898 249 173 032 979 797 454 822 091 361 516 493 685 200 519 168
65	332 064 020 377 997 870 516 666 134 986 911 324 093 053 522 016 491	332 064 020 377 997 870 525 666 748 294 395 962 690 152 372 895 744
66	1 992 384 122 267 987 223 126 998 649 843 954 096 284 137 490 727 674	1 992 384 122 267 987 223 154 000 489 766 375 776 140 914 237 374 464
67	10 711 714 176 249 442 775 842 490 621 525 929 005 859 791 304 709 265	10 711 714 176 249 442 775 915 076 020 156 878 430 286 136 158 978 048
68	72 227 515 851 910 895 072 546 510 984 548 819 851 947 461 500 127 744	72 227 515 851 910 895 072 791 227 327 048 682 674 446 154 223 058 944
69	484 504 539 895 354 302 547 182 982 411 390 143 288 400 081 568 182 252	484 504 539 895 354 302 548 003 765 028 162 025 351 465 729 976 696 832

k	$q_k(n_k)$	n_k
70	2 580 045 416 677 653 746 793 287 325 520 051 094 567 582 723 781 309 860	2 580 045 416 677 653 746 795 472 708 993 390 547 711 242 654 533 550 080
71	15 481 983 506 619 376 167 450 475 756 940 595 595 391 083 576 728 101 855	15 481 983 506 619 376 167 457 032 632 000 074 795 982 120 027 168 964 608
72	79 724 435 061 469 256 513 775 209 426 621 989 319 474 734 478 061 022 890	79 724 435 061 469 256 513 792 091 732 421 567 953 656 961 220 772 626 432
73	44 635 565 825 867 831 030 166 773 394 546 228 926 061 036 211 284 733 894	44 635 565 825 867 831 030 171 499 368 905 460 606 154 328 014 352 652 697
74	3 703 344 512 426 048 548 703 143 823 299 003 324 157 052 495 257 779 500 644	3 703 344 512 426 048 548 703 339 876 725 069 484 658 804 966 586 383 859 712
75	21 518 289 689 844 262 782 547 057 828 694 599 081 556 375 014 229 295 503 138	21 518 289 689 844 262 782 547 627 413 080 094 761 002 777 135 389 903 683 584

Table 2

$$1 - \exp - \left(\frac{7}{48}\,(108)^k (E_k)^2\right) = H(k)$$

k	H(k)	k	H(k)	k	H(k)	k	H(k)
5	.424	23	.084	41	.054	59	.047
6	.136	24	.168	42	.003	60	.151
7	.862	25	.824	43	.022	61	.799
8	.658	26	.444	44	.390	62	.587
9	.634	27	.498	45	.729	63	.020
10	.007	28	.393	46	.813	64	.243
11	.047	29	.420	47	.252	65	.206
12	.665	30	.506	48	.923	66	.932
13	.879	31	.163	49	.603	67	.433
14	.540	32	.030	50	.170	68	.318
15	.356	33	.107	51	.766	69	.989
16	.671	34	.207	52	.514	70	.306
17	.290	35	.943	53	.137	71	.039
18	.232	36	.679	54	.352	72	.903
19	.573	37	.938	55	.999	73	.052
20	.591	38	.910	56	.866	74	.324
21	.770	39	.688	57	.446	75	.004
22	.717	40	.589	58	.642		

Table 3

$n_k/6^k$

k	$n_k/6^k$	k	$n_k/6^k$	k	$n_k/6^k$	k	$n_k/6^k$
6	.672	24	.654	42	.778	60	.928
7	.844	25	.964	43	.973	61	.773
8	.699	26	.567	44	.526	62	.819
9	.775	27	.713	45	.501	63	.939
10	.807	28	.758	46	.931	64	.740
11	.807	29	.727	47	.966	65	.873
12	.785	30	.960	48	.995	66	.873
13	.934	31	.835	49	.793	67	.782
14	.856	32	.956	50	.792	68	.879
15	.778	33	.529	51	.876	69	.983
16	.919	34	.613	52	.967	70	.873
17	.901	35	.551	53	.888	71	.873
18	.826	36	.957	54	.987	72	.749
19	.782	37	.801	55	.691	73	.699
20	.912	38	.514	56	.695	74	.966
21	.543	39	.768	57	.991	75	.936
22	.760	40	.963	58	.952		
23	.950	41	.671	59	.583		

Table 4

Values for $n_{k,r}$ and $q_{k,r}(n_{k,r})$ for $2 \leq r \leq 20$

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
2	3	100	73
	4	28	18
	5-7	28	17
	8+	176	106
3	4	8	7
	5-6	27	23
	7	378	316
	8	136	113
	9+	378	314
4	5-6	16	15
	7	96	89
	8	81	75
	9	976	903
	10	656	606
	11	1296	1197
	12+	2512	2320

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
5	6-7	32	31
	8	243	235
	9	256	251
	10	736	710
	11	1216	1173
	12+	3168	3055
6	7-9	64	63
	10	768	755
	11	1472	1447
	12-13	3648	3586
	14	16064	15790
	15+	31360	30825
7	8-11	128	127
	12	2304	2285
	13	6656	6601
	14	15360	15233
	15	24064	23865
	16	126848	125799
	17	78848	78195
	18+	236288	234331

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
8	9-12	256	255
	13-14	6656	6629
	15-16	19712	19632
	17	124672	124166
	18	781250	778080
	19-20	393728	392129
	21+	1174528	1169758
9	10-14	512	511
	15	19968	19928
	16-17	39424	39345
	18-19	177152	176797
	20	2086400	2082219
	21	1988096	1984111
	22	3936768	3928877
	23+	7814151	7798488
10	11-15	1024	1023
	16-17	59392	59333
	18-22	177152	176976
	23	12164096	12152011
	24	9802752	9793012
	25	29347840	29318680
	26+	48833536	48785015

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
11	12-17	2048	2047
	18	178176	178088
	19-24	354304	354129
	25	97785856	97737557
	26	49070080	49045842
	27	97785856	97737555
	28	195393536	195297022
	29+	293001216	292856489
12	13-19	4096	4095
	20-21	532480	532349
	22-25	2125824	2125301
	26	37732352	37723069
	27	108945408	108918605
	28	244465664	244405519
	29-30	488394752	488274594
	31+	1709645824	1709225206

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
13	14–20	8192	8191
	21–22	1597440	1597244
	23	4784128	4783541
	24–25	12754944	12753379
	26	46235648	46229975
	27	113197056	113183167
	28	146677760	146659763
	29	326836224	326796122
	30	1487503360	1487320847
	31	1221255168	1221105322
	32	3663757312	3663307775
	33	8545574912	8544526384
	34+	12207734784	12206236915
14	15–22	16384	16383
	23–26	4784128	4783835
	27	71745536	71741142
	28	205668352	205655756
	29–30	473513984	473484984
	31–32	1487503360	1487412259
	33	6107856896	6107482824
	34	12210929664	12210181813
	35	24417058816	24415563408
	36+	67143319552	67139207400

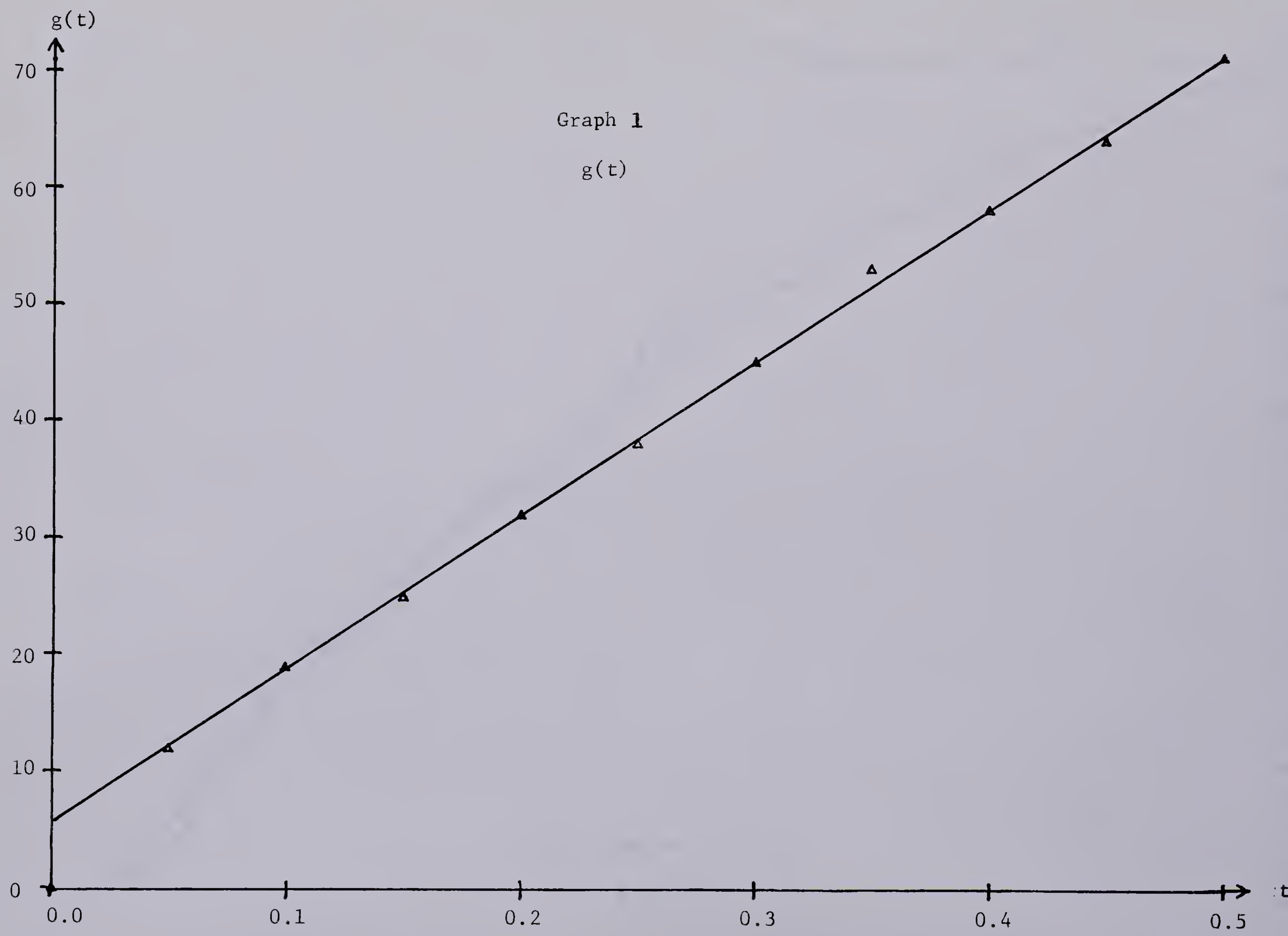
r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
15	16-23	32768	32767
	24-27	14352384	14351945
	28	143491072	143486683
	29	272629760	272621421
	30-32	947027968	946999001
	33-34	4462510080	4462373584
	35	30563172352	30562237507
	36	61040263168	61038396111
	37	91560378368	91557577783
	38+	183106404352	183100803621
16	17-25	65536	65535
	26-28	43057152	43056494
	29	301334528	301329923
	30	817889280	817876781
	31-33	1894055936	1894026991
	34	14334558208	14334339147
	35-36	26775060480	26774651303
	37	785129144320	785117145961
	38	152600641536	152598309489
	39	457887973376	457880975920
	40	762960084992	762948425416
	41	1678477754368	1678452103827
	42+	2593995423744	2593955782238

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
17	18-26	131072	131071
	27	129236992	129236005
	28	258342912	258340939
	29	387448832	387445873
	30	904003584	903996680
	31	1937113088	1937098294
	32	2453667840	2453649101
	33	8394113024	8394048917
	34-37	14334558208	14334448733
	38	160650362880	160649135971
	39	1623808409600	1623796008351
	40	763089256448	763083428629
	41	1525920169984	1525908516319
	42-43	3051840339968	3051817032638
	44+	15258814251008	15258697717317

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
18	19-28	262144	262143
	29-31	387448832	387447353
	32	3874226176	3874211387
	33	7361003520	7360975421
	34-39	14334558208	14334503489
	40	892229386240	892225980352
	41	1770124214272	1770117457215
	42	3814929727488	3814915164838
	43-44	7629471744000	7629442620180
	45	26702957248512	26702855315882
	46	64849928257536	64849680707286
	47+	83923413762048	83923093402988

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
19	20-30	524288	524287
	31-32	1162346496	1162344278
	33	8135901184	8135885659
	34	15109455872	15109427040
	35	29056565248	29056509802
	36	43003674624	43003592564
	37-38	86007349248	86007185128
	39	301025722368	301025147948
	40	559047770112	559046703332
	41-43	1770124214272	1770120836505
	44	137768662990848	137768400099442
	45	19077359730688	19077323327097
	46	38147745906688	38147673112813
	47	114442075373568	114441856994161
	48	267030734307328	267030224756857
	49+	476837525323776	476836615418082

r	k	$n_{k,r}$	$q_{k,r}(n_{k,r})$
20	21-31	1048576	1048575
	32	3487563776	3487560449
	33	6974078976	6974072323
	34	10460594176	10460584197
	35	24407703552	24407680268
	36	66249031680	66248968481
	37	122037469184	122037352765
	38-40	258022047744	258021801601
	41	1942138912768	1942137060043
	42	3626255777792	3626252318485
	43-45	5310372642816	5310367576927
	46	199593999466496	199593809061582
	47	95380987445248	95380896455494
	48	190737567186944	190737385230720
	49	286108094038016	286107821102017
	50	762939808153600	762939080338683
	51	2098085343789056	2098083342297203
	52+	3337860352573440	3337857168384426

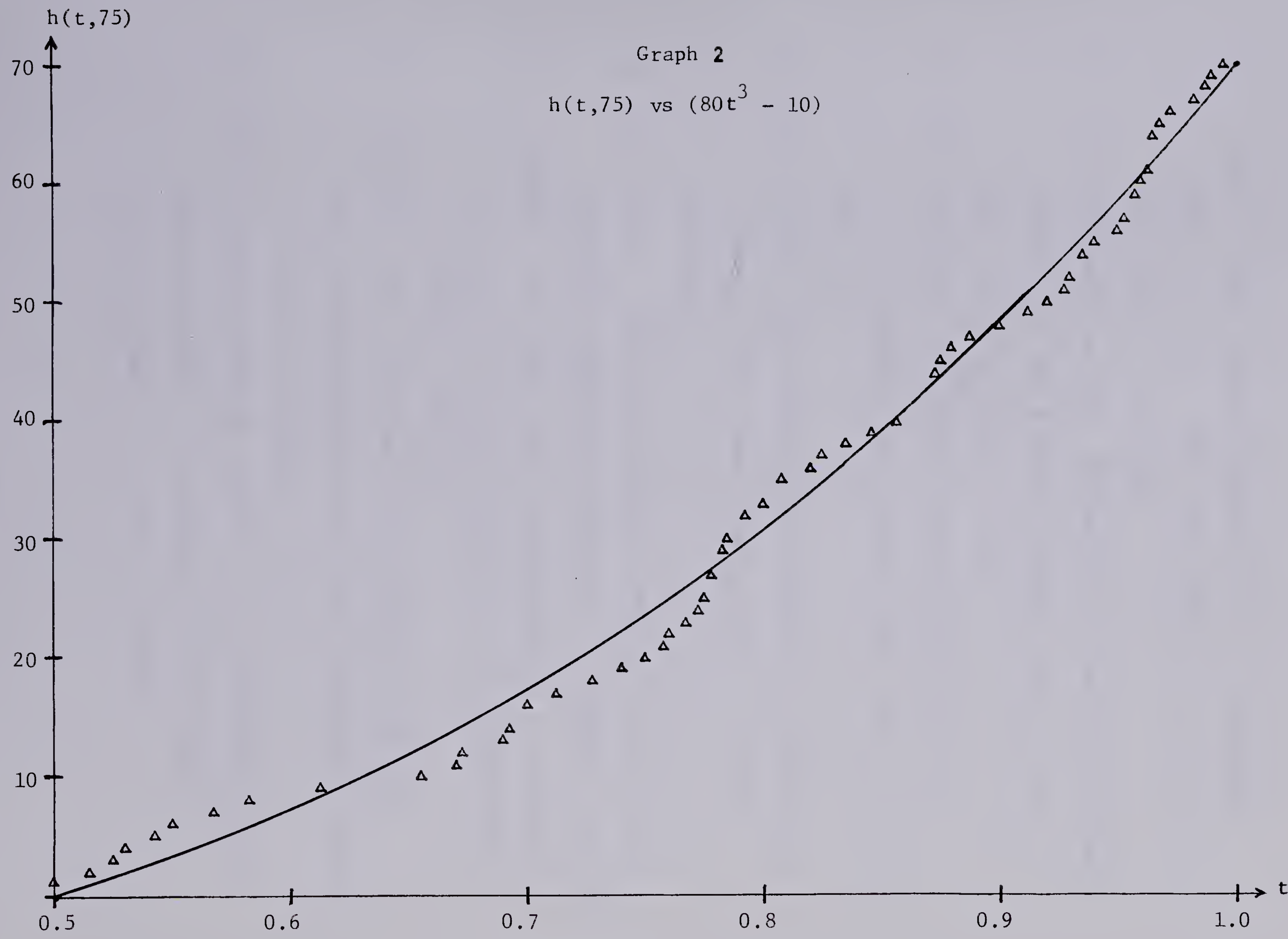


Graph 1

g(t)

Graph 2

$h(t,75)$ vs $(80t^3 - 10)$



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